Optimal Communication Rates and Combinatorial Properties of Distributed Simulation

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Joint work with:

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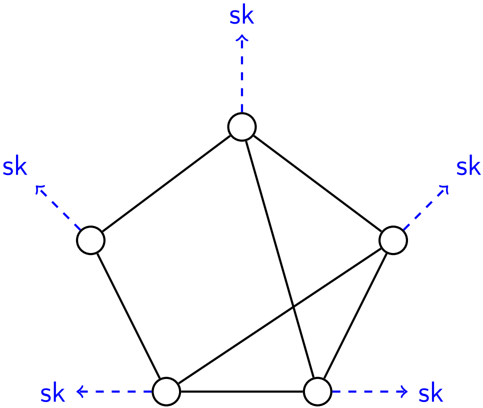
WOLA 2020

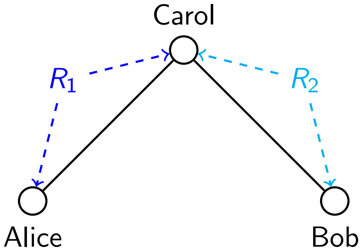
Generation of Shared Randomness



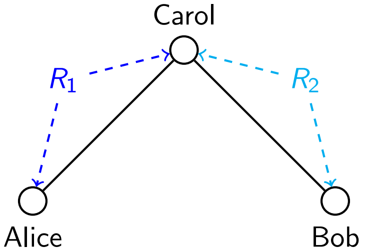
Secret key generation

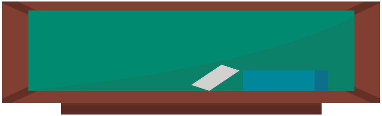
Other applications: distributed computing, distributed inference, game theory, quantum mechanics...

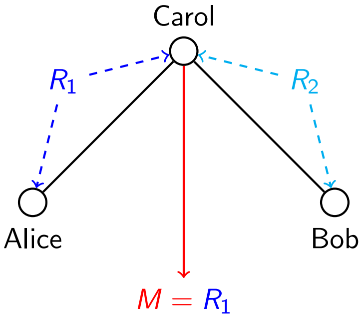
Toy Example



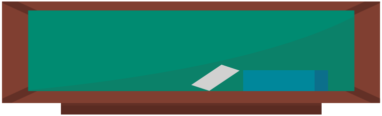
Toy Example



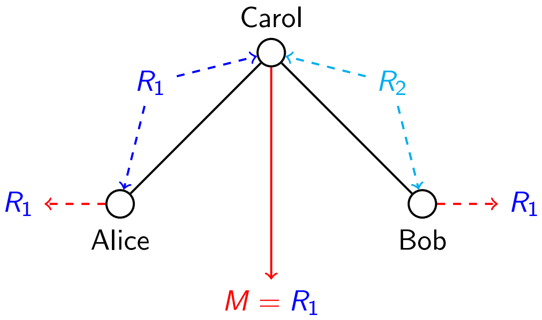


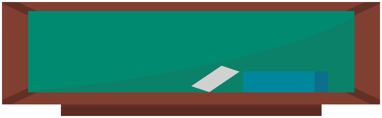


Carol
R_1
R_2
Alice
Bob
$M = R_1$

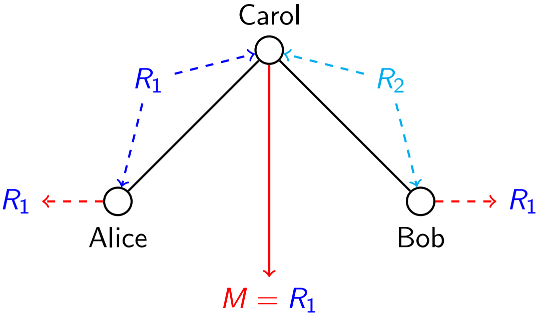


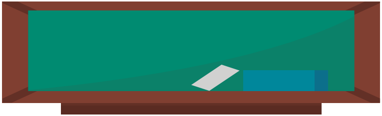
Toy Example





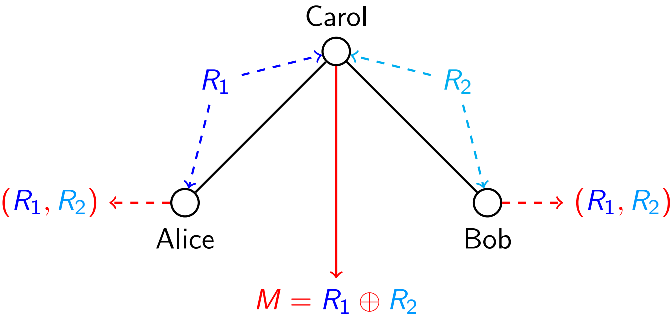
Toy Example

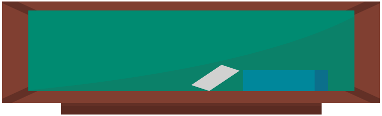




Communication rate $= 1$.

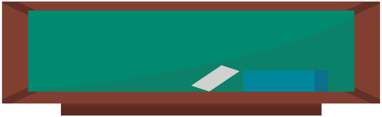
Toy Example



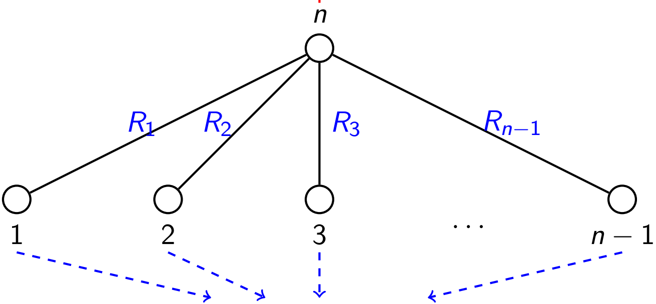


Communication rate $= 1/2$.

Generalization to n Users



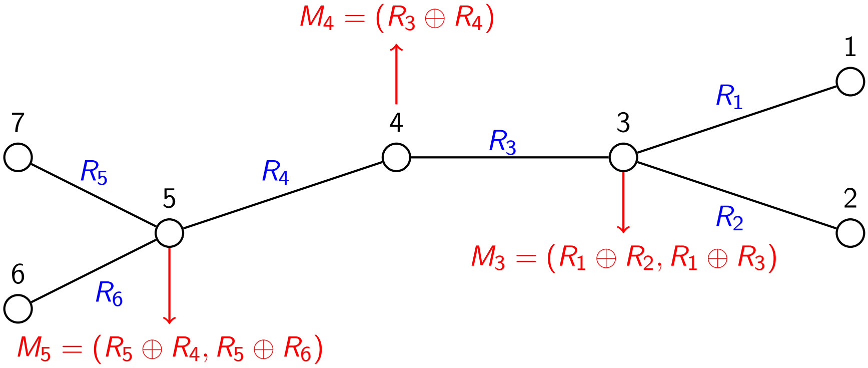
$$M = (R_1 \oplus R_2, R_1 \oplus R_3, \cdots, R_1 \oplus R_{n-1})$$



$$R = (R_1, \cdots, R_{n-1})$$

Communication rate $= |M|/|R| = (n-2)/(n-1)$.

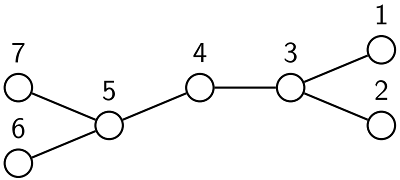
Generalization to Connected Graphs

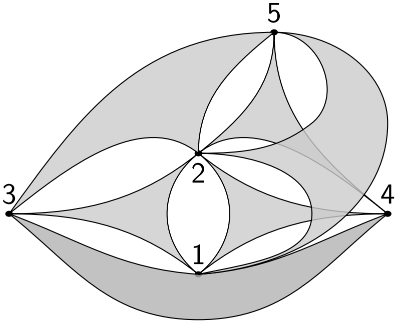


$$\text{Communication rate} = \frac{5}{6} = \frac{7-2}{7-1}.$$

Problem Formulation

- Let G be a hypergraph with vertex set $[n]$ and edge set consisting of all groups of users who share common randomness





- Under the blackboard communication procotol, users write message M on the blackboard
- Each of the user simulates the same random sequence R
- Target: find the minimum communication rate $|R|/|M|$

Motivating Result

Theorem

For n users and k-complete hypergraph G, the optimal communication rate is $(n-k)/(n-1)$.

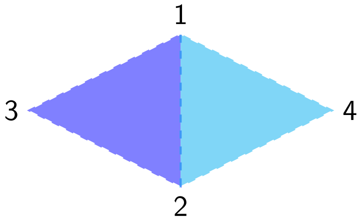
- lower bounds the communication rate for any k-uniform hypergraph
- the previous rate $(n-2)/(n-1)$ is optimal for $k=2$
- connected graphs suffice for $k=2$
- tight result for general hypergraph is available, but in terms of a computationally intractable linear program

Target

Find proper connectivity notions for general k-uniform hypergraphs.

Generalization of Connectivity

- Usual connectivity notions for hypergraphs break down here
- For example, the communication rate $1/3$ cannot be achieved in the following hypergraph:



- Idea: generalize the following folklore:

Folklore

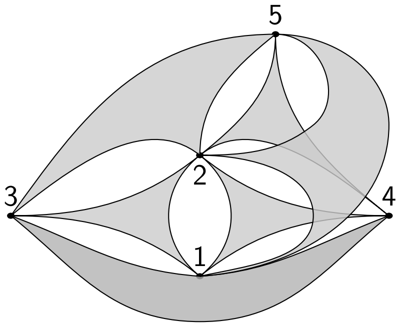
Any tree on n vertices has $n-1$ edges.

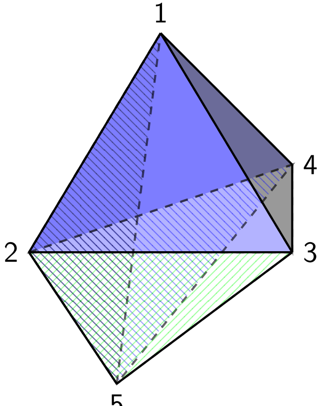
- Answer: topological connectivity and path connectivity

Topological Connectivity

Definition (Topological Connectivity, Kalai'83)

A k-uniform hypergraph is topologically k-connected iff it becomes the complete graph after adding the last missing facet of each k-dim polygon finitely many times.





Combinatorial Property

Theorem

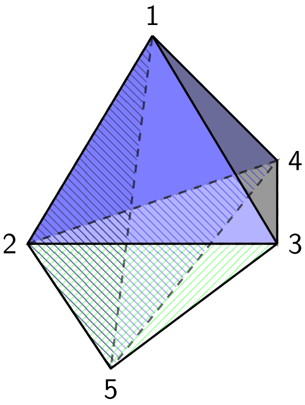
Any minimal topologically k-connected hypergraph has $\binom{n-1}{k-1}$ hyperedges.

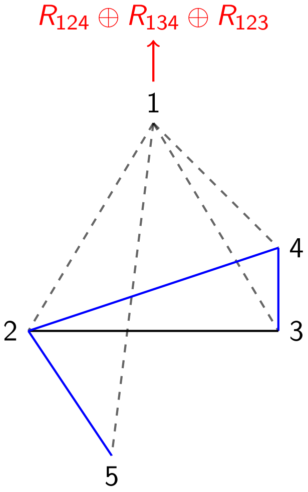
- First proof via Euler's formula $\sum_{j=0}^{k-1}(-1)^j F_j = \chi = 1$ with $F_j = \binom{n}{j+1}$ for $0 \leq j \leq k-2$, and $F_{k-1} = |E|$
- Second proof via incident matrix of G:

$$
\begin{array}{c}
 \\ (123) \\ (124) \\ (134) \\ (125) \\ (235) \\ (245)
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc}
(12) & (13) & (14) & (15) & (23) & (24) & (25) & (34) & (35) & (45)
\end{array} \\
\begin{bmatrix}
1 & 1 & & & 1 & & & & & \\
1 & & 1 & & & 1 & & & & \\
 & 1 & 1 & & & & & 1 & & \\
1 & & & 1 & & & 1 & & & \\
 & & & & 1 & & 1 & & 1 & \\
 & & & & & 1 & 1 & & & 1
\end{bmatrix}
\end{array}
$$

with full row rank and number of rows being $|E|$

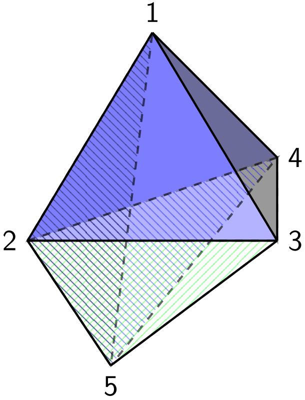
Communication Strategy

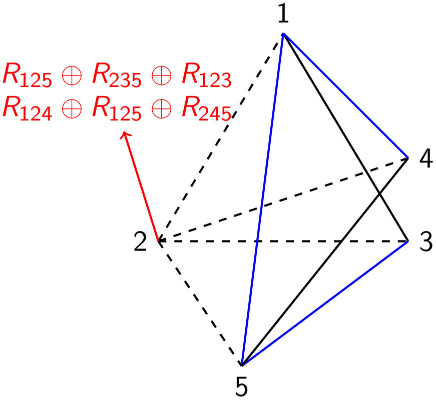




$M = R_{124} \oplus R_{134} \oplus R_{123}$, $\quad R = (R_{123}, R_{124}, R_{125}, R_{134}, R_{235}, R_{245})$.

Communication Strategy





$$M = \begin{cases} R_{124} \oplus R_{134} \oplus R_{123} \\ R_{125} \oplus R_{235} \oplus R_{123} \\ R_{124} \oplus R_{125} \oplus R_{245} \end{cases}$$

$$R = (R_{123}, R_{124}, R_{125}, R_{134}, R_{235}, R_{245}).$$

Analysis

- Bits of shared randomness:

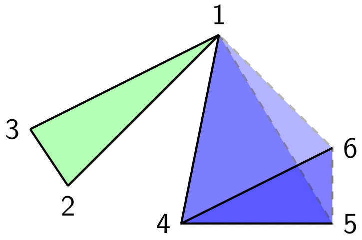
$$|R| = \binom{n-1}{k-1}.$$

- Bits of messages:

$$|M| = \sum_{i=1}^{n} \left(d_i - \binom{n-2}{k-2} \right) = k\binom{n-1}{k-1} - n\binom{n-2}{k-2} = \binom{n-2}{k-1}.$$

- Decodability: see full paper

A Counterexample



- Not topologically 3-connected
- However, admit a strategy acheving optimal communication rate 3/5:

$$R = (R_{123}, R'_{123}, R_{145}, R_{146}, R_{456})$$

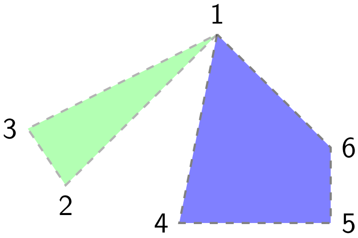
$$M = \begin{cases} R_{123} \oplus R_{145} \\ R'_{123} \oplus R_{146} \\ R_{145} \oplus R_{146} \oplus R_{456} \end{cases}$$

- Repetition required

Path Connectivity

Definition (Path Connectivity)

A hypergraph $G = (V, \{E_1, \cdots, E_m\})$ is path connected iff for every $u, v \in V$, there exist a sequence $v_0 = u, v_1, \cdots, v_{n-1}, v_n = v$ such that for every $i \in [n]$, we have $\{v_{i-1}, v_i\} \subseteq E_j$ for some $j \in [m]$.



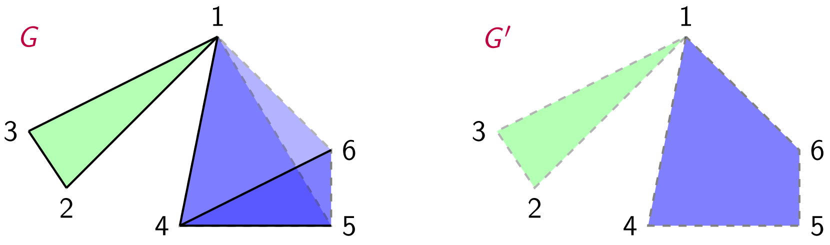
Theorem

If $G = ([n], \{E_1, \cdots, E_m\})$ is path-connected and cycle-free, then $\sum_{j=1}^{m}(|E_j| - 1) = n - 1$.

Path-connected Cycle-free Cluster

Definition

A uniform k-hypergraph G is a path-connected cycle-free cluster of topologically connected components iff there exists another hypergraph $G' = (V, \{E_1, \cdots, E_m\})$ which is path-connected and cycle-free, and the restriction of G to each E_j is topologically k-connected.



Theorem

The optimal communication rate $(n-k)/(n-1)$ is achievable for path-connected cycle-free clusters of topologically connected components.

Concluding Remarks

Take-home messages:

- Two generalizations of connectivity and the tree folklore
- Both generalizations give local algorithms for distributed simulation

Thank you!
Arxiv: 1904.03271