Batched Multi-armed Bandits Problem

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Introduction

Stochastic multi-armed bandits problem:

- Arms of a stochastic bandit $\mathcal{I} = \{1, 2, \cdots, K\}$, $K \geq 2$.
- Reward of pulling arm i at time t: $r_{t,i} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu^{(i)}, 1)$.
- Time horizon T.
- A predictable process $\pi = (\pi_t)_{t=1}^T$ with regard to the filtration $\mathcal{F}_t = \{A_1, A_2, \cdots, A_t, r_{1,A_1}, r_{2,A_2}, \cdots, r_{t,A_t}\}$.

Batch constraints:

- Grid of M batches, $1 \leq t_1 < t_2 < \cdots < t_M = T$.
- For $t_j < t \leq t_{j+1}$, π_t is $\mathcal{F}_{t_j}$ measurable.

Two types of grid:

- Static grid: Fix the grid beforehand.
- Adaptive grid: Determine t_{j+1} based on $\mathcal{F}_{t_j}$.

Target: Minimize regret

$$R_T(\pi) \triangleq \sum_{t=1}^{T} \left(\mu^\star - \mu^{(\pi_t)}\right) = T\mu^\star - \sum_{t=1}^{T} \mu^{(\pi_t)},$$

under batched constraints, where $\mu^\star = \max_{i \in [K]} \mu^{(i)}$.

Two Types of Regrets

We aim to characterize the following *minimax regret* and *problem-dependent regret* under the batched setting:

$$R^\star_{\text{min-max}}(K, M, T) \triangleq \inf_{\pi \in \Pi_{M,T}} \sup_{\{\mu^{(i)}\}_{i=1}^K : \Delta_i \leq \sqrt{K}} \mathbb{E}[R_T(\pi)],$$

$$R^\star_{\text{pro-dep}}(K, M, T) \triangleq \inf_{\pi \in \Pi_{M,T}} \sup_{\Delta > 0} \Delta \cdot \sup_{\{\mu^{(i)}\}_{i=1}^K : \Delta_i \in \{0\} \cup [\Delta, \sqrt{K}]} \mathbb{E}[R_T(\pi)].$$

where $\Pi_{M,T}$ is the set of policies with M batches and horizon T, and $\Delta_i = \mu^\star - \mu^{(i)}$.

Related Works

Without batch constraint [1, 2]:

$$R^\star_{\text{min-max}}(K, T, T) = \Theta(\sqrt{KT}),$$
$$R^\star_{\text{pro-dep}}(K, T, T) = \Theta(K \log T).$$

Required number of batches [3]:

$$R^\star_{\text{min-max}}(K, \log\log T, T) = \widetilde{\Theta}(\sqrt{KT}).$$

Two-armed bandit with static grid [4]:

$$R^\star_{\text{min-max}}(2, M, T) = \widetilde{\Theta}(T^{1/(2-2^{1-M})}),$$
$$R^\star_{\text{pro-dep}}(2, M, T) = \widetilde{\Theta}(T^{1/M}).$$

Main Results

Theorem 1 (Upper Bound): There exist policies π^1, π^2 such that

$$\mathbb{E}[R(\pi^1)] \leq \text{polylog}(K, T) \cdot \sqrt{K} T^{\frac{1}{2-2^{1-M}}},$$
$$\mathbb{E}[R(\pi^2)] \leq \text{polylog}(K, T) \cdot \frac{KT^{\frac{1}{M}}}{\min_{i \neq \star} \Delta_i}.$$

Theorem 2 (Static Lower Bound): Under any static grid,

$$R_{\text{min-max}}(K, M, T) = \Omega(\sqrt{K} T^{\frac{1}{2-2^{1-M}}}),$$
$$R_{\text{pro-dep}}(K, M, T) = \Omega(KT^{\frac{1}{M}}).$$

Theorem 3 (Adaptive Lower Bound): Under any adaptive grid,

$$R_{\text{min-max}}(K, M, T) = \Omega(M^{-2} \cdot \sqrt{K} T^{\frac{1}{2-2^{1-M}}}),$$
$$R_{\text{pro-dep}}(K, M, T) = \Omega(M^{-2} \cdot KT^{\frac{1}{M}}).$$

Remark:

- It is sufficient to have $M = O(\log\log T)$ batches to achieve the optimal minimax regret $\Theta(\sqrt{KT})$, and $M = O(\log T)$ to achieve the optimal problem-dependent regret $\Theta(K \log T)$.
- With either static or adaptive grids, it is necessary to have $M = \Omega(\log\log T)$ batches to achieve the optimal minimax regret $\Theta(\sqrt{KT})$, and $M = \Omega(\log T / \log\log T)$ to achieve the optimal problem-dependent regret $\Theta(K \log T)$.
- It is an open problem to remove the M^{-2} factor in the adaptive lower bound.

BaSE Policy

Key Idea: Sequentially drop the arms which are "significantly" worse than the "best" one.

BaSE (Batched Successive Elimination)

Input: number of arms K, number of batches M, time horizon T, time grid $\mathcal{T}$, tuning parameter $\gamma > 0$

Output: policy π

initialize the set of active arms $\mathcal{A} \leftarrow [K]$;

for $m = 1$ to M **do**

pull all active arms for same number of times in m-th batch;

for $i \in \mathcal{A}$ **do**

compute the mean reward $\bar{r}_i$ for arm i;

end for

compute the maximum mean reward $r_{\max} = \max_{i \in \mathcal{A}} \bar{r}_i$ and the number of pullings τ_m for each active arm;

eliminate all active arms with $r_{\max} - \bar{r}_i \geq \sqrt{\gamma \log(TK)/\tau_m}$ from $\mathcal{A}$;

end for

Optimal Grid Design

Minimax grid: $t_1 = a, t_m = \lfloor a\sqrt{t_{m-1}} \rfloor$, where $a = \Theta\left(T^{1/(2-2^{1-M})}\right)$.

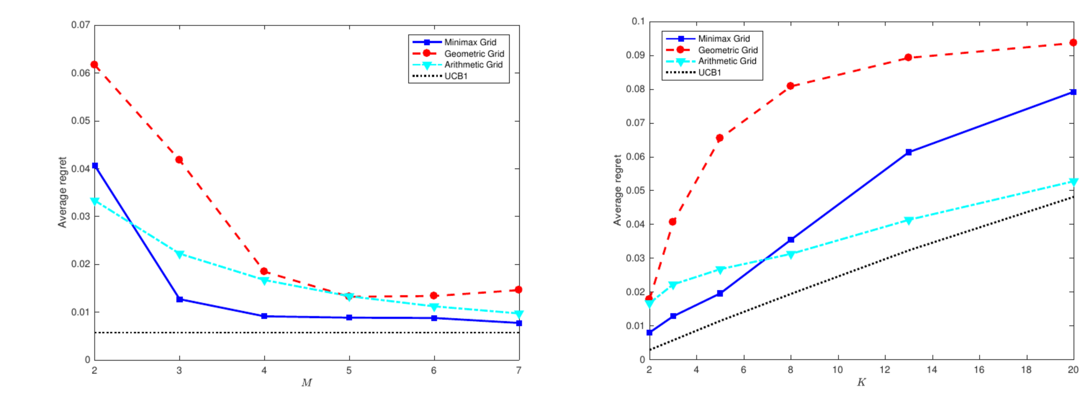
Geometric grid: $t'_1 = b, t'_m = \lfloor a t'_{m-1} \rfloor$, where $b = \Theta\left(T^{1/M}\right)$.

Numerical Experiments

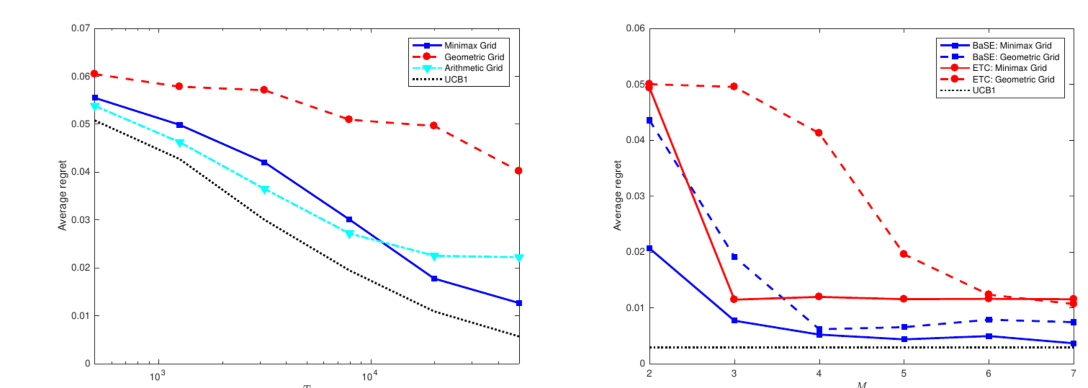
Setting:

- Parameters: $T = 5 \times 10^4$, $K = 3$, $M = 3$ and $\gamma = 1$.
- Mean reward: $\mu^\star = 0.6$ for the optimal arm and $\mu = 0.5$ for all other arms.
- Implement minimax grid, geometric grid and the arithmetic grid with $t_j = jT/M$ for $j \in [M]$.
- Baseline: UCB1 algorithm [5] without any batch constraints.

Experimental results:



(a) Average regret vs. the number of batches M. (b) Average regret vs. the number of arms K.



(c) Average regret vs. the time horizon T. (d) Comparison of BaSE and ETC.

Observations:

- The minimax grid typically results in a smallest regret among all grids.
- $M = 4$ batches appear to be sufficient for the BaSE performance to approach the centralized performance.

References

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