# A Cap-Move Reformulation of Ling's Proof of Samuels' Conjecture

Yanjun Han[*]

August 22, 2026

**Abstract**

Let $0 \le \mu_1 \le \mu_2 \le \cdots \le \mu_n$ and $\delta > 0$. Samuels' conjecture claims that if $X_1, \ldots, X_n$ are independent non-negative random variables with $\mathbb{E}[X_i] = \mu_i$, then

$$\mathbb{P}\left(\sum_{i=1}^n X_i < \delta + \sum_{i=1}^n \mu_i\right) \ge \min_{1\le i\le n} \prod_{j=i}^n \left(1 - \frac{\mu_j}{\delta + \sum_{k=i}^n \mu_k}\right).$$

This conjecture was recently proved by Ling [Lin26]. In this note, we provide an alternative presentation of Ling's proof via an operation called the *cap move*. This proof works directly with finitely supported distributions and avoids the reduction to the Bernoulli case.

## 1 Introduction

Let $X_1, \ldots, X_n$ be independent non-negative random variables with $\mathbb{E}[X_i] = \mu_i$, with $0 \le \mu_1 \le \mu_2 \le \cdots \le \mu_n$. Samuels' conjecture [Sam66, Sam68, Sam69] claims the following:

**Conjecture 1** (Samuels' Conjecture). *For every $\delta > 0$,*

$$\mathbb{P}\left(\sum_{i=1}^n X_i < \delta + \sum_{i=1}^n \mu_i\right) \ge \min_{1\le i\le n} \prod_{j=i}^n \left(1 - \frac{\mu_j}{\delta + \sum_{k=i}^n \mu_k}\right). \tag{1}$$

Samuels' conjecture implies several other conjectures such as Feige's [Fei04, Fei06], as well as a number of results in extremal combinatorics [AFH+12, FJ19].

Very recently Ling proved Samuels' conjecture [Lin26]. Ling's proof first reduces the problem to the Bernoulli case where each $X_i$ takes the form of $w_i\mathrm{Bern}(p_i)$, and then uses an induction argument by showing that a suitable coordinate can be removed one at a time. The current note provides a cap-move formulation of Ling's idea that works for general finitely supported distributions.

**AI disclosure.** This note benefits from discussions with GPT Sol 5.6 Pro, but is written entirely by the author.

## 2 Proof of Samuels' Conjecture

By approximating a general random variable with discrete ones, it suffices to consider the case where each $X_i$ is a discrete random variable with finite support. In fact, a simple convexity argument can reduce the support size to 2, but the subsequent proof does not require this stronger result.

[*]New York University, yanjunhan@nyu.edu

## 2.1 The Cap Move

A key ingredient of the proof is the following *cap move*: pick some $i \in [n]$ and $a \geq 0$, replace $X_i$ by $X_i \wedge a := \min\{X_i, a\}$. Let $p$ be the LHS probability of (1), and $p_i^a$ be the counterpart after applying the cap move. Since a cap move makes the sum $\sum_{i=1}^n X_i$ no larger, we have $p_i^a \geq p$. The following lemma controls the increase from $p$ to $p_i^a$.

**Lemma 2.1** (Cap Move)**.** *Let $S = \sum_{i=1}^n X_i$ with $\mathbb{E}[S] < \lambda$, and $S_i^a$ be the counterpart after applying the cap move $X_i \mapsto X_i \wedge a$. Also define $p = \mathbb{P}(S < \lambda), p_i^a = \mathbb{P}(S_i^a < \lambda)$, and*

$$b(S) := \mathbb{E}[S] - \mathbb{E}\left[S \mid S < \lambda\right], \tag{2}$$

$$r(S) := \lambda - \mathbb{E}\left[S \mid S < \lambda\right]. \tag{3}$$

*Then $b(S) \geq b(S_i^a)$, and*

$$\frac{p_i^a}{p} \leq \frac{r(S)}{r(S) - (b(S) - b(S_i^a))}.$$

*Proof.* Write $S_{-i} = \sum_{j \neq i} X_j$, which is independent of $X_i$. Then

$$\begin{aligned}
\mathbb{E}[(\lambda - S_i^a)_+] - \mathbb{E}[(\lambda - S)_+] &= \mathbb{E}[\mathbf{1}_{\{X_i > a\}}[(\lambda - S_{-i} - a)_+ - (\lambda - S_{-i} - X_i)_+]] \\
&\leq \mathbb{E}[\mathbf{1}_{\{X_i > a\}}(X_i - a)\mathbf{1}_{\{\lambda - S_{-i} - a > 0\}}] \\
&= \mathbb{E}[\mathbf{1}_{\{X_i > a\}}(X_i - a)] \cdot \mathbb{P}(S_{-i} + a < \lambda) \\
&\leq \mathbb{E}[(X_i - a)_+] \cdot p_i^a.
\end{aligned}$$

Since $\mathbb{E}[(\lambda - S)_+] = pr(S)$, $\mathbb{E}[(\lambda - S_i^a)_+] = p_i^a r(S_i^a)$, and

$$r(S) - r(S_i^a) = b(S) - b(S_i^a) - \mathbb{E}[(X_i - a)_+], \tag{4}$$

rearranging gives the target inequality. The claim follows from this inequality and $p_i^a \geq p$. $\square$

By Lemma 2.1, for a cap move that does not increase $p$ by too much, we want a large $r(S)$ and a small $b(S) - b(S_i^a)$. This motivates the following notion of *safe move*.

**Definition 2.2** (Safe Move)**.** *We call a cap move* safe *if $r(S_i^a) \geq r(S)$.*

Before we proceed, we comment on the consequences of a safe move. First, by definition, the quantity $r(S)$ is non-decreasing under safe moves. Second, by (4), a safe move implies $b(S) - b(S_i^a) \leq \mathbb{E}[(X_i - a)_+] = \mathbb{E}[X_i] - \mathbb{E}[X_i \wedge a]$; therefore, after any finite sequence of safe moves, the sum of such terms with index $i$ is at most $\mathbb{E}[X_i] = \mu_i$. Finally, over a finite sequence of safe moves, the sum of all $b$-decrements is at most $b(S)$. These properties turn out to be crucial in the final proof.

## 2.2 Existence of Safe Moves

The second step is a key claim in the proof: unless all $X_i$'s are deterministic, a non-trivial safe move always exists. This follows from the following lemma.

**Lemma 2.3.** *Assume the conditions and notations of Lemma 2.1. For each $i \in [n]$, there exists a nonnegative measure $\nu_i$ on $\mathbb{R}_+$ such that*

$$\mathsf{Cov}(S, X_i \mid S < \lambda) = \int_0^\infty (r(S_i^a) - r(S))\nu_i(da).$$

*In addition, if $\operatorname{supp}(X_i) = \{x_1, \ldots, x_m\}$ with $x_1 < \cdots < x_m$, then $\operatorname{supp}(\nu_i) = \{x_1, \ldots, x_{m-1}\}$.*

*Proof.* A simple proof uses an exponential tilt. For $\theta \geq 0$, let $P_i$ be the distribution of $X_i$, and $P_{i,\theta}(dx_i) \propto e^{-\theta x_i} P_i(dx_i)$ be its exponential tilt. Also write $S_{i,\theta}$ be the counterpart of $S$ when only $P_i$ is replaced by $P_{i,\theta}$. By standard properties of exponential tilts,

$$\frac{d}{d\theta} r(S_{i,\theta})\Big|_{\theta=0} = \mathsf{Cov}(S, X_i \mid S < \lambda),$$
$$\frac{d}{d\theta} P_{i,\theta}(dx)\Big|_{\theta=0} = (m_i - x) P_i(dx),$$

with $m_i := \mathbb{E}[X_i]$. To relate the LHS quantities, note that

$$r(S) = \lambda - \mathbb{E}[S \mid S < \lambda] = \frac{\mathbb{E}[(\lambda - S)_+]}{\mathbb{E}[\mathbf{1}_{\{S<\lambda\}}]}$$
$$\implies \frac{d}{d\theta} r(S_{i,\theta})\Big|_{\theta=0} = \int \frac{d}{d\theta} P_{i,\theta}(dx_i)\Big|_{\theta=0} \cdot \frac{\mathbb{E}[(\lambda - S_{-i} - x_i)_+] - r(S)\mathbb{P}(S_{-i} + x_i < \lambda)}{\mathbb{P}(S < \lambda)}.$$

We next show that there exists a measure $\nu_{i,0}$ such that

$$(m_i - x) P_i(dx) = \int_0^\infty \nu_{i,0}(da)(P_i^a - P_i)(dx), \tag{5}$$

where $P_i^a$ is the distribution of $X_i \wedge a$. To find $\nu_{i,0}$, we integrate both sides on $[0, t]$ for $t \geq 0$:

$$\begin{aligned}
\mathbb{E}[(X_i - m_i)\mathbf{1}_{\{X_i > t\}}] &= \int_{(t,\infty)} (x - m_i) P_i(dx) \\
&= \int_{[0,t]} (m_i - x) P_i(dx) \\
&= \int_0^\infty \nu_{i,0}(da)(P_i^a - P_i)([0, t]) \\
&= \int_0^\infty \nu_{i,0}(da)\mathbb{P}(X_i > t)\mathbf{1}_{\{a \leq t\}} = \mathbb{P}(X_i > t)\nu_{i,0}([0, t]).
\end{aligned}$$

Therefore, we can construct $\nu_{i,0}(dt) = dw_i(t)$, with $w_i(t) := \mathbb{E}[X_i - m_i \mid X_i > t]$. Clearly $w_i(0^-) = 0$ and $t \mapsto w_i(t)$ is non-decreasing, so $\nu_{i,0}$ is a nonnegative Stieltjes measure. Using (5) and the above displays, we have

$$\mathsf{Cov}(S, X_i \mid S < \lambda) = \int_0^\infty \nu_{i,0}(da) \frac{\mathbb{P}(S_i^a < \lambda)}{\mathbb{P}(S < \lambda)} (r(S_i^a) - r(S)),$$

and absorbing the (strictly positive) ratio into $\nu_{i,0}$ completes the proof of the identity. The support follows from $\mathrm{supp}(\nu_i) = \mathrm{supp}(\nu_{i,0}) = \{x_1, \ldots, x_{m-1}\}$ and the jump structure of the Stieltjes measure $dw_i(t)$. □

The following corollary is then immediate.

**Corollary 2.4** (Existence of a safe move)**.** *Let $X_1, \ldots, X_n$ be discrete with finite support, and not all deterministic. Then there exist $i \in [n]$ and $a \in \mathrm{supp}(X_i), 0 \leq a < \|X_i\|_\infty$ such that the cap move $X_i \mapsto X_i \wedge a$ is safe.*

*Proof.* Summing Lemma 2.3 over $i \in [n]$ gives

$$0 \leq \mathrm{Var}(S \mid S < \lambda) = \sum_{i=1}^n \int_0^\infty (r(S_i^a) - r(S))\nu_i(da).$$

As a result, there exist $i \in [n]$ and $a \in \mathrm{supp}(\nu_i)$ with $r(S_i^a) \geq r(S)$, i.e. the cap move $X_i \mapsto X_i \wedge a$ is safe. By the second statement of Lemma 2.3, we have $a \in \mathrm{supp}(X_i)$ and $0 \leq a < \|X_i\|_\infty$. □

### 2.3 Variational representation of Samuels' lower bound

The final ingredient is a variational representation of Samuels' lower bound.

**Lemma 2.5** (Variational representation). *For $0 \le \mu_1 \le \cdots \le \mu_n$ and $\delta > 0$, it holds that*

$$\min_{b\ge 0} \min_{\substack{z_i\in[0,\mu_i], i\in[n] \\ z_1+\cdots+z_n\le b}} \sum_{i=1}^n \log\left(1-\frac{z_i}{\delta+b}\right) = \min_{1\le i\le n} \sum_{j=i}^n \log\left(1-\frac{\mu_j}{\delta+\sum_{k=i}^n \mu_k}\right).$$

*Proof.* Let $M_k = \sum_{i=k}^n \mu_i$ for $k\in[n]$, and $M_{n+1} := 0$. For a fixed $b\in[M_{k+1}, M_k]$ with $k\ge 1$, since $z\mapsto \log(1-\frac{z}{\delta+b})$ is decreasing and concave, an inner minimizer is $z_i = \mu_i$ for $i\ge k+1$, $z_i = 0$ for $i<k$, and $z_k = b - M_{k+1}$. Therefore, for $b\in[M_{k+1}, M_k]$,

$$G(b) := \min_{\substack{z_i\in[0,\mu_i], i\in[n] \\ z_1+\cdots+z_n\le b}} \sum_{i=1}^n \log\left(1-\frac{z_i}{\delta+b}\right) = \log\left(1-\frac{b-M_{k+1}}{\delta+b}\right) + \sum_{i=k+1}^n \log\left(1-\frac{\mu_i}{\delta+b}\right).$$

On the current interval,

$$G'(b) = \frac{1}{\delta+b}\left[-1+\sum_{i=k+1}^n \frac{\mu_i}{\delta+b-\mu_i}\right],$$

and the term in the bracket is decreasing in $b$. Therefore, $G'(b)$ can change sign at most once, and only from positive to negative; this shows that $G$ is quasi-concave and $G(b)\ge G(M_k)\wedge G(M_{k+1})$ on $[M_{k+1}, M_k]$. This implies

$$\min_{0\le b\le M_1} G(b) = \min_{1\le i\le n} G(M_i) = \min_{1\le i\le n}\sum_{j=i}^n \log\left(1-\frac{\mu_j}{\delta+\sum_{k=i}^n \mu_k}\right).$$

The case $b\ge M_1$ is obvious: the sum constraint is inactive, and $G(b)$ is increasing in $b\ge M_1$. $\square$

### 2.4 Completing the proof

We are ready to complete the proof of Conjecture 1. Start from $S_0 := S = \sum_{i=1}^n X_i$, and assume that all $X_i$'s are discrete with finite support. By Corollary 2.4, as long as $X_i$'s are not all deterministic, there exists a safe move $X_i \mapsto X_i \wedge a$, and repeating such operations gives a sequence $S_0, S_1, S_2, \ldots$. By Corollary 2.4, each cap move will remove one support from $X_i$, so the finite support assumption implies that these are only finitely many safe moves, and the final sum $S_m$ is deterministic. Set $\lambda = \delta + \sum_{i=1}^n \mu_i$ and let $p_t = \mathbb{P}(S_t < \lambda)$; since $\mathbb{E}[S_t]$ is non-increasing, we have $S_m = \mathbb{E}[S_m] \le \sum_{i=1}^n \mu_i < \lambda$, so $p_m = 1$. Also, the definition of safe moves implies $r(S_0)\le r(S_1)\le\cdots\le r(S_m)$, so Lemma 2.1 gives

$$\frac{p_t}{p_{t-1}} \le \frac{r(S_0)}{r(S_0)-(b(S_{t-1})-b(S_t))}.$$

Let $z_i = \sum_{t\in[m]:i_t=i}(b(S_{t-1})-b(S_t))$, with $i_t$ being the coordinate capped at time $t$, the discussion below Definition 2.2 gives

$$0\le z_i\le \mu_i, \qquad \sum_{i=1}^n z_i \le b(S_0).$$

Finally, a telescoping gives

$$\begin{aligned}
\log \mathbb{P}(S_0 < \lambda) &\geq \log \mathbb{P}(S_m < \lambda) + \sum_{i=1}^{n} \sum_{t\in[m]:i_t=i} \log\left(1 - \frac{b(S_{t-1}) - b(S_t)}{r(S_0)}\right) \\
&\geq \sum_{i=1}^{n} \log\left(1 - \frac{z_i}{r(S_0)}\right) = \sum_{i=1}^{n} \log\left(1 - \frac{z_i}{\delta + b(S_0)}\right) \\
&\geq \min_{b\geq 0} \min_{\substack{z_i\in[0,\mu_i],i\in[n] \\ z_1+\cdots+z_n\leq b}} \sum_{i=1}^{n} \log\left(1 - \frac{z_i}{\delta + b}\right).
\end{aligned}$$

The variational representation in Lemma 2.5 then proves Samuels' conjecture.

## References


[AFH+12] Noga Alon, Peter Frankl, Hao Huang, Vojtech Rödl, Andrzej Ruciński, and Benny Sudakov. Large matchings in uniform hypergraphs and the conjectures of Erdős and Samuels. *Journal of Combinatorial Theory, Series A*, 119(6):1200–1215, 2012.

[Fei04] Uriel Feige. On sums of independent random variables with unbounded variance, and estimating the average degree in a graph. In *Proceedings of the Thirty-Sixth Annual ACM Symposium on Theory of Computing*, pages 594–603, 2004.

[Fei06] Uriel Feige. On sums of independent random variables with unbounded variance and estimating the average degree in a graph. *SIAM Journal on Computing*, 35(4):964–984, 2006.

[FJ19] Asaf Ferber and Vishesh Jain. Uniformity-independent minimum degree conditions for perfect matchings in hypergraphs. *arXiv preprint arXiv:1903.12207*, 2019.

[Lin26] Zhi Ling. On Samuels' conjecture. *arXiv preprint arXiv:2608.18392*, 2026.

[Sam66] Stephen M. Samuels. On a Chebyshev-type inequality for sums of independent random variables. *Annals of Mathematical Statistics*, 37(1):248–259, 1966.

[Sam68] Stephen M. Samuels. More on a Chebyshev-type inequality for sums of independent random variables. Technical Report 155, Department of Statistics, Purdue University, 1968.

[Sam69] Stephen M. Samuels. The Markov inequality for sums of independent random variables. *Annals of Mathematical Statistics*, 40(6):1980–1984, 1969.