

An Empirical Bayes Perspective on Heteroskedastic Mean Estimation

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Abstract

Towards understanding the fundamental limits of estimation from data of varied quality, we study the problem of estimating a mean parameter from heteroskedastic Gaussian observations where the variances are unknown and may vary arbitrarily across observations. While a simple linear estimator with known variances attains the smallest mean squared error, estimation without this knowledge is challenging due to the large number of nuisance parameters. We propose a simple and principled approach based on empirical Bayes: model the observations as if they were i.i.d. from a normal scale mixture and compute the profile maximum likelihood estimator (MLE) for the mean, treating the nonparametric mixing distribution as nuisance. Our result shows that this estimator achieves near-optimal error bounds across various heteroskedastic models in the literature. In particular, for the subset-of-signals problem where an unknown subset of observations has small variance, our estimator adaptively achieves the minimax rate for all signal sizes, including the sharp phase transition, without any tuning parameters.

One of our key technical steps is a sharper metric entropy bound for normal scale mixtures, obtained via Chebyshev approximations on a transformed polynomial basis. This approach yields an improved polylogarithmic, rather than polynomial, dependence on the variance ratio, which could be of independent interest.

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1 Introduction

Estimation of a signal from heterogeneous data lies at the heart of statistics and machine learning. The heterogeneity can arise from variations in data quality, measurement precision, or sampling conditions and is ubiquitous in real-world applications. Thus, it is a fundamental statistical challenge to develop principled methods that allow one to estimate signals robustly from heterogeneous data with minimal knowledge of the data quality.

Perhaps the simplest abstraction of this general objective is estimation of a single one dimensional parameter from data of varying quality. In particular, given independent observations X_1, \dots, X_n with $X_i \sim \mathcal{N}(\mu, \sigma_i^2)$, the learner’s target is to estimate the mean parameter $\mu \in \mathbb{R}$. Even in this simple one dimensional setting, heterogeneity leads to statistical challenges. In the case when $(\sigma_1, \dots, \sigma_n)$ are known, the maximum likelihood estimator (MLE) for μ is

$$\hat{\mu}_{\text{known-}\sigma} = \frac{\sum_{i=1}^n \frac{X_i}{\sigma_i^2}}{\sum_{i=1}^n \frac{1}{\sigma_i^2}}, \tag{1}$$

which is also the uniformly minimum-variance unbiased estimator (UMVUE) and achieves the optimal error rate of $\mathbb{E}[|\hat{\mu} - \mu|] = (\sum_{i=1}^n \frac{1}{\sigma_i^2})^{-1/2}$. The key aspect of this estimator is that this bound is *robust* to large variances. For example, if there are $n/2$ variances that are large while the other $n/2$ variances are small, then the error rate is still $O(1/\sqrt{n})$, as the large variances are effectively ignored in the weighted average. Compare this to the error of the sample average $\bar{\mu} = \frac{1}{n} \sum_{i=1}^n X_i$,

which has error rate $\mathbb{E}[(\bar{\mu} - \mu)^2] = \frac{1}{n^2} \sum_{i=1}^n \sigma_i^2$ and could be arbitrarily large even if a single variance is large. On the other hand, $\hat{\mu}_{\text{known-}\sigma}$ critically relies on the knowledge of $(\sigma_1, \dots, \sigma_n)$, which could not be estimated reliably simply because the number of unknown parameters $n + 1$ is more than the number of observations n while the sample average $\bar{\mu}$ is oblivious to the knowledge of $(\sigma_1, \dots, \sigma_n)$. The key question that we would like to address is:

Are there general principled methods that allow one to estimate signals robustly from heterogeneous data with minimal knowledge of the data quality?

An important special case considered in the literature is the “subset of signals” problem [LY20], where it is assumed that $|\{i \in [n] : \sigma_i \leq 1\}| \geq m$, i.e., m out of n observations can be treated as “signals” (although the location of this subset of signals is still unknown). Even for this simple setting, the minimax rate has only been recently characterized [CV24]: there exists an estimator $\hat{\mu}$ with

$$\mathbb{E}|\hat{\mu} - \mu| = \begin{cases} \tilde{O}\left(\left(\frac{n}{m^4}\right)^{1/2}\right) & \text{if } \log n \ll m \leq n^{1/4}, \\ \tilde{O}\left(\left(\frac{n}{m^4}\right)^{1/6}\right) & \text{if } n^{1/4} \leq m \leq n, \end{cases} \quad (2)$$

and this dependence on (m, n) (including the phase transition at $m \asymp n^{1/4}$) is not improvable in the worst case.¹ While the minimax rate is known, existing estimators [CV24] achieving this minimax rate are often quite complicated, requiring multiple tunable parameters and are reliant on the particular formulation of the problem, and tend not to lead to unifying statistical principles for learning from heterogeneous data.

Towards building a unified theory of heteroskedasticity in estimation, we make a perhaps surprising connection to the study of *empirical Bayes* methods. Using this connection, we show that the simple principle of maximum likelihood estimation, in conjunction with an empirical Bayes framework, leads to a unified estimator that achieves the minimax rate adaptively for all ranges of m . In this framework, instead of modeling the heterogeneous variances $\sigma_1, \dots, \sigma_n$ separately, we model them using their empirical distribution $G_n := \frac{1}{n} \sum_{i=1}^n \delta_{\sigma_i}$ and treat the observations X_1, \dots, X_n as if they were *i.i.d.* drawn from a normal scale mixture $f_{\mu, G_n}(x) := \mathbb{E}_{\sigma \sim G_n}[\frac{1}{\sigma} \varphi(\frac{x-\mu}{\sigma})]$, where φ denotes the standard normal density. Under this *i.i.d.* model, the joint MLE $(\hat{\mu}, \hat{G})$ is naturally defined as

$$(\hat{\mu}, \hat{G}) = \arg \max_{\mu \in \mathbb{R}, \text{supp}(G) \subseteq [\sigma_{\min}, \sigma_{\max}]} \frac{1}{n} \sum_{i=1}^n \log f_{\mu, G}(X_i), \quad (3)$$

and the final estimator is simply defined as $\hat{\mu}^{\text{EB}} = \hat{\mu}$ in (3). In other words, our estimator $\hat{\mu}^{\text{EB}}$ is the profile MLE [MVdV00] of μ , treating G as a nuisance parameter. Note that the nuisance estimator \hat{G} in (3) is a *nonparametric* MLE (NPMLE), since it may be any probability distribution supported on $[\sigma_{\min}, \sigma_{\max}]$. In particular, this formulation no longer enforces the structure of an empirical distribution, nor does it explicitly require the “subset-of-signals” structure $G([0, 1]) \geq \frac{m}{n}$. Here $\sigma_{\min}, \sigma_{\max} > 0$ are hyperparameters used in the MLE and satisfy $\sigma_{\min} \leq \sigma_i \leq \sigma_{\max}$ for every $i \in [n]$; further discussion on them are provided later.

¹Here and throughout, $\tilde{O}(\cdot)$ hides polylogarithmic factors in n .

1.1 Main results

Our first result shows that, although the estimator $\hat{\mu}^{\text{EB}}$ is developed from an entirely different principle compared with existing estimators, it nevertheless has an instance-dependent error bound.

Theorem 1.1. *Let $\sigma_i \in [\sigma_{\min}, \sigma_{\max}]$ for all $i \in [n]$. With probability at least $1 - \delta$, the estimator $\hat{\mu}^{\text{EB}}$ in (3) achieves (the exact logarithmic factor is displayed in [Theorem 1.3](#))*

$$|\hat{\mu}^{\text{EB}} - \mu| \leq C\omega_{H^2, G_n} \left(\frac{\text{polylog}(n, \frac{\sigma_{\max}}{\sigma_{\min}}, \frac{1}{\delta})}{n} \right),$$

where $\omega_{H^2, G_n}(t)$ is the Hellinger modulus of continuity in the location family:

$$\omega_{H^2, G_n}(t) = \sup \{ |\mu_1 - \mu_2| : \mu_1, \mu_2 \in \mathbb{R}, H^2(f_{\mu_1, G_n}, f_{\mu_2, G_n}) \leq t \}.$$

We remark that [Theorem 1.1](#) establishes an upper bound that is competitive with the best oracle estimator possessing knowledge of G_n , uniformly over all possible choices of G_n . Indeed, even in the mean estimation problem where X_1, \dots, X_n are i.i.d. drawn from f_{μ, G_n} with *known* G_n , Le Cam's two-point method [[LC73](#), [LC86](#)] yields a minimax lower bound of $\omega_{H^2, G_n}(\frac{1}{n})$.² By comparison, the radius in the Hellinger modulus of continuity in our upper bound is $\tilde{O}(\frac{1}{n})$, which essentially matches this lower bound even when G_n is unknown.

Specializing to several heterogeneous mean estimation models studied in the literature [[PJL22](#), [DLLZ23](#)], [Theorem 1.1](#) implies the following error bounds, all of which match the best known rates ([PJL22](#), Table 1] and [[DLLZ23](#), Section 5.1]).

Corollary 1.1. *The following high-probability guarantee holds in specific examples:*

1. *Equal variance: when $\sigma_i \equiv 1$, then $|\hat{\mu}^{\text{EB}} - \mu| = \tilde{O}(\frac{1}{\sqrt{n}})$;*
2. *Quadratic variance: when $\sigma_i = i$, then $|\hat{\mu}^{\text{EB}} - \mu| = \tilde{O}(1)$;*
3. *Two variances: when $\sigma_1 = \dots = \sigma_m = 1$ and $\sigma_{m+1} = \dots = \sigma_n = \sigma \geq 1$ with $m \leq \frac{n}{2}$, then $|\hat{\mu}^{\text{EB}} - \mu| = \tilde{O}(\frac{\sigma}{\sqrt{n}})$. For specific ranges of m , sharper upper bounds can be obtained:*

$$|\hat{\mu}^{\text{EB}} - \mu| = \begin{cases} \tilde{O}(\frac{\sqrt{n}}{m}) & \text{if } m \geq \sqrt{n \log n}, \\ O(1) & \text{if } \sqrt{\frac{n \log n}{\sigma}} + \log n \leq m < \sqrt{n \log n}. \end{cases}$$

4. *α -mixture distribution: when $\sigma_1 = \dots = \sigma_m = 1$ and $\sigma_{m+1} = \dots = \sigma_n = n^\alpha$ with $m = c \log n$, then $|\hat{\mu}^{\text{EB}} - \mu| = \tilde{O}(n^{\alpha-1/2})$ if $0 < \alpha < 1$ and $|\hat{\mu}^{\text{EB}} - \mu| = O(1)$ if $\alpha \geq 1$.*

Finally, for the subset-of-signals problem with m signals, the estimator $\hat{\mu}^{\text{EB}}$ achieves near-optimal error bounds, including the correct phase transition, across all ranges of m :

Theorem 1.2. *Let $\sigma_i \in [\sigma_{\min}, \sigma_{\max}]$ for all $i \in [n]$, and the number of signals be m in the subset-of-signals problem. With probability at least $1 - \delta$, estimator $\hat{\mu}^{\text{EB}}$ in (3) achieves*

$$|\hat{\mu}^{\text{EB}} - \mu| = \begin{cases} \tilde{O}\left(\left(\frac{n}{m^4}\right)^{1/2}\right) & \text{if } \log n \ll m \leq n^{1/4}, \\ \tilde{O}\left(\left(\frac{n}{m^4}\right)^{1/6}\right) & \text{if } n^{1/4} \leq m \leq n, \end{cases}$$

where $\tilde{O}(\cdot)$ hides polylogarithmic factors in $(n, \frac{\sigma_{\max}}{\sigma_{\min}}, \frac{1}{\delta})$.

²This precise argument does not hold in the compound setting, but this intuition is conjectured to remain valid. See [Section 5](#) for detailed discussions.

Compared with the known minimax risk (2) for the subset-of-signals problem, [Theorem 1.2](#) shows that the estimator $\hat{\mu}^{\text{EB}}$ achieves the minimax rate up to logarithmic factors under the mild assumption $\log \frac{\sigma_{\max}}{\sigma_{\min}} = O(\text{polylog}(n))$. However, like many empirical Bayes approaches, a particularly appealing advantage of our estimator is that it is *adaptive* and *parameter-free*: this estimator requires no tuning parameters (we set $\sigma_{\min} \approx 0$ and σ_{\max} to be large in our experiments; see [Section 4](#)), and adapts to the signal size m . These features make $\hat{\mu}^{\text{EB}}$ especially attractive in practice. In the practical scenario where the joint MLE (3) is computed only approximately, a similar guarantee remains valid for approximate MLEs; see [Remark 1](#).

Compared with existing empirical Bayes approaches, while the idea of applying an NPMLE-based estimator is classical in the empirical Bayes literature [[KW56](#), [Rob56](#)], our objective in (3) is conceptually different. In existing theoretical studies of empirical Bayes, one typically considers independent observations $X_i \sim P_{\theta_i}$ (the compound setting) and aims to estimate the parameter vector $(\theta_1, \dots, \theta_n)$. The empirical Bayes approach proceeds by first estimating the empirical distribution $G_n := \frac{1}{n} \sum_{i=1}^n \delta_{\theta_i}$ (e.g., via the NPMLE), and then applying the resulting learned prior to estimate θ through the corresponding Bayes rule. By contrast, our procedure likewise replaces heterogeneity in the *nuisance parameter* $(\sigma_1, \dots, \sigma_n)$ with a mixing distribution G_n and learns it from the data, but does not invoke explicit posterior inference. It is therefore an interesting theoretical observation that empirical Bayes methods are near-optimal in our setting, even though the procedure does not actually invoke a ‘‘Bayes’’ component.

We also provide some discussions on the hyperparameters $(\sigma_{\min}, \sigma_{\max})$ and computation. We acknowledge that our estimator requires additional knowledge of $(\sigma_{\min}, \sigma_{\max})$, which is not needed by previous estimators. This requirement is, however, intrinsic to our approach: without it, the solution to (3) would be degenerate, since one could assign a positive mass to $\hat{G}(\{0\})$ and choose $\hat{\mu} \in \{X_1, \dots, X_n\}$, thereby making the likelihood unbounded. Fortunately, the error dependence on σ_{\min} in [Theorem 1.2](#) is only $\text{polylog}(\frac{\sigma_{\max}}{\sigma_{\min}})$ and thus very mild, and our numerical experiments show that even setting $\sigma_{\min} = 0, \sigma_{\max} = \infty$ still yields sound estimation performance. As for computation, solving the optimization program in (3) is, unfortunately, a nonconvex problem involving the infinite-dimensional parameter \hat{G} . In [Section 4](#), we adopt a successive maximization strategy. For fixed μ , (3) reduces to a convex optimization problem in G , for which it is standard to apply a fully corrective Frank–Wolfe algorithm. For fixed G , (3) becomes a one-dimensional maximization problem in μ , which can be efficiently solved via grid search. Although both Frank–Wolfe and successive maximization may converge to local maxima, our numerical results indicate that the resulting $\hat{\mu}^{\text{EB}}$ is empirically close to the ideal MLE obtained when G_n is known exactly.

1.2 Outline of the proof

[Theorem 1.1](#) follows from a key result concerning the density estimation performance of the MLE, which accurately estimates the true mixing density f_{μ, G_n} in Hellinger distance.

Theorem 1.3. *Let $\sigma_{\min} \leq \sigma_i \leq \sigma_{\max}$ for all $i \in [n]$, and $(\hat{\mu}, \hat{G})$ be the joint MLE in (3). Then with probability at least $1 - \delta$,*

$$H^2(f_{\mu, G_n}, f_{\hat{\mu}, \hat{G}}) \leq \frac{C}{n} \left(\log^3 \left(\frac{n\sigma_{\max}}{\sigma_{\min}} \right) \log^4 \log \left(\frac{n\sigma_{\max}}{\sigma_{\min}} \right) + \log \frac{1}{\delta} \right).$$

Remark 1. *If $(\hat{\mu}, \hat{G})$ is an approximate MLE with $\prod_{i=1}^n f_{\hat{\mu}, \hat{G}}(X_i) \geq \beta \cdot \max_{\mu, G} \prod_{i=1}^n f_{\mu, G}(X_i)$ with $\beta \in (0, 1]$, by [Lemma 2](#), the Hellinger bound in [Theorem 1.3](#) has an additive factor $O(\frac{1}{n} \log \frac{1}{\beta})$.*

[Theorem 1.3](#) establishes an upper bound in Hellinger distance for estimating the density f_{μ, G_n} , which is the true average distribution of X_1, \dots, X_n . Obtaining density estimation guarantees under

the Hellinger metric is standard in the statistical literature for analyzing the MLE, thanks to its well-known connection with the metric entropy of the underlying density class [WS95, vdG00]. [Theorem 1.1](#) is then a direct consequence of [Theorem 1.3](#) and a symmetrization inequality for the Hellinger distance in [Lemma 8](#), presented in [Section B](#).

Upper bound on the covering number In the proof of [Theorem 1.3](#), a key technical step is to analyze the metric entropy of the family of normal scale mixtures $\{f_{0,G} : \text{supp}(G) \subseteq [\sigma_{\min}, \sigma_{\max}]\}$ under the Hellinger metric. Although such normal scale mixtures have been studied in the literature [[GvdV01](#), [GvdV07](#), [IS25](#)], the existing arguments based on (local) moment matching [[GvdV01](#), [Zha09](#)] yield metric entropy bounds with a dependence of $\text{poly}(\frac{\sigma_{\max}}{\sigma_{\min}})$; by contrast, we critically need to improve this dependence to $\text{polylog}(\frac{\sigma_{\max}}{\sigma_{\min}})$. To achieve this, we introduce two simple yet effective ideas.

- First, instead of applying classical moment matching based on polynomials, we perform a generalized moment matching by constructing a suitable basis $\{a_k(\sigma), g_k(x)\}$ to approximate the normal density $\frac{1}{\sigma}\varphi(\frac{x}{\sigma})$ by a separable expansion $\sum_{k=1}^L a_k(\sigma)g_k(x)$.
- Second, instead of using the usual Taylor approximating polynomial for a given analytic function, we employ Chebyshev polynomials, which more accurately capture growth behavior on a Bernstein ellipse rather than on a disk in the complex plane.

These arguments establish the $\text{polylog}(\frac{\sigma_{\max}}{\sigma_{\min}})$ dependence, which we elaborate in [Section 2](#).

Upper bounding Hellinger modulus of continuity To prove [Theorem 1.2](#), we need to upper bound the Hellinger modulus of continuity $\omega_{H^2,G}(t)$. We prove such a bound in the case when $G([0, 1]) \geq p$ for some $p > 0$, which captures instances such as the subset-of-signals problem.

Lemma 1. *Let $t \in [0, 1]$, and G be a prior distribution over $[0, \infty)$ with $G([0, 1]) \geq p > 0$. Then for a universal constant $C > 0$,*

$$\omega_{H^2,G}(t) \leq C \begin{cases} (\frac{t^3}{p^4})^{1/6} & \text{if } t \leq p^{4/3}, \\ (\frac{t^3}{p^4})^{1/2} & \text{if } p^{4/3} < t \leq \frac{p}{C}. \end{cases} \quad (4)$$

[Lemma 1](#) is a functional inequality that exploits the structure of normal scale mixtures and will be proved in [Section D.1](#). We note that this inequality is not improvable in general, as witnessed by the example $G = p\delta_1 + (1-p)\delta_{c\mu/\sqrt{t}}$, with $\mu = \mu(p, t)$ matching the upper bound in [Lemma 1](#); we omit the algebra verifying this claim. The condition $t = O(p)$ is also necessary, as convexity of squared Hellinger distance applied to the above choice of G yields

$$H^2(f_{-\mu,G}, f_{\mu,G}) \leq 2p + (1-p)H^2(\mathcal{N}(-\mu, \frac{c^2\mu^2}{t}), \mathcal{N}(\mu, \frac{c^2\mu^2}{t})) \leq t$$

regardless of the choice of μ , as long as $t \geq 3p$ and c is a large universal constant. Therefore, when $t \geq 3p$, we have $\omega_{H^2,G}(t) = \infty$. We note that [Theorem 1.2](#) follows directly from [Theorem 1.1](#) and [Lemma 1](#), with the subset-of-signals assumption giving $G_n([0, 1]) \geq \frac{m}{n} =: p$. [Corollary 1.1](#) similarly follows from [Theorem 1.1](#) and specific calculations of $\omega_{H^2,G_n}(t)$ for the respective choices of G_n , which we defer to [Section D.2](#).

1.3 Related Work

We provide a brief review of the relevant literature here and defer the full discussion to [Section A](#).

Heteroskedastic mean estimation. The problem of estimating a common location parameter from heterogeneous observations has attracted significant attention in statistics, theoretical computer science and econometrics. Versions of the general problem have been studied under names such as entangled means [CDKL14] and subset-of-signals [LY20]. In particular, for this setting [CV24] provides a minimax characterization of the rates. [DKLP25] obtained an efficient algorithm for the high dimensional analogue of the problem. Returning to the general heteroskedastic estimation problem, [PJL22] characterize minimax rates for location estimation under sample-heterogeneous distributions and propose estimators based on iterative trimming and median-of-means techniques. [DLLZ23] further developed this line of work, providing a more fine-grained analysis of achievable rates. [Nar25] obtained a sharp characterization of rates achievable as a function of the variances. In addition, for normal scale mixtures with known G_n , an upper bound in terms of the Hellinger modulus of continuity was established in [CV25, Theorem 1.6] using a different Birgé–Le Cam type estimator. Our empirical Bayes perspective shows that the profile MLE attains a comparable bound even when G_n is unknown.

Empirical Bayes methods. Empirical Bayes methodology, pioneered by Robbins [Rob51, Rob56], provides a principled framework for estimation problems with many latent parameters in the compound decision setting. Here, one observes $X_i \sim P_{\theta_i}$ for $i = 1, \dots, n$ and aims to estimate the parameter vector $(\theta_1, \dots, \theta_n)$. The empirical Bayes approach treats the parameters as draws from an unknown prior G and estimates G from the marginal distribution of the observations. From both theoretical [PW21] and practical [ETST01] perspectives, empirical Bayes has become a successful framework for large-scale statistics with the nonparametric maximum likelihood estimator (NPMLE) for the mixing distribution, introduced by Kiefer and Wolfowitz [KW56], playing a central role in this framework. The most widely studied problems in the literature have been the Normal location model [Zha03, Zha09, BG09, SG20, PW21, GIKL25] and the Poisson model [BGR13, PW21, SW22, JPTW23, JPW25, HNWSW25]. See [Efr24] and [IS25] for a survey.

The conceptual view on empirical Bayes is that it models heterogeneity in the data using a mixture distribution (prior) and learns the prior from data. From this perspective, heteroskedastic mean estimation is a natural test bed for empirical Bayes methodology and this is the motivation for our work. However, we emphasize a major difference between our application of the methodology from the standard applications: while standard empirical Bayes would have treated the heterogeneous parameters $(\sigma_1, \dots, \sigma_n)$ as the primary estimation target, we treat them as nuisance parameters and focus on estimating the common mean μ through a profile likelihood [MVdV00, Sev00]. This perspective connects our work to the literature on profile likelihood inference with infinite-dimensional nuisance parameters [MVdV97].

2 Density Estimation: Proof of Theorem 1.3

In this section we prove the density estimation guarantee of the MLE in Theorem 1.3. We begin with a general entropic upper bound of the Hellinger distance in the compound (or i.n.i.d.) setting, and then present a metric entropy upper bound for the normal scale mixture with a fixed mean. The remaining details will be devoted to allowing μ to vary; we defer them to the appendix.

2.1 Entropic Upper Bound of the MLE in the Compound Setting

In this section we review and state the classical entropic upper bound for density estimation via the MLE, in a general i.n.i.d. (independent and *non-identically* distributed) setting. Let X_1, \dots, X_n

be independent, with $X_i \sim P_i$. Given a class \mathcal{P} of distributions such that $\bar{P} := \frac{1}{n} \sum_{i=1}^n P_i \in \mathcal{P}$, let \hat{P} be a β -approximate MLE with $\prod_{i=1}^n \hat{P}(X_i) \geq \beta \cdot \max_{P \in \mathcal{P}} \prod_{i=1}^n P(X_i)$. To bound the Hellinger distance $H(\hat{P}, \bar{P})$, we need the following notations for the metric entropy. For $\delta > 0$, define the localized family $\mathcal{P}(\bar{P}, \delta) := \{P \in \mathcal{P} : H(P, \bar{P}) \leq \delta\}$.

Definition 1 (Bracketing Number). *The (δ -local) Hellinger bracketing number at scale ε around \bar{P} , denoted by $N_{[]}(\varepsilon, \mathcal{P}(\bar{P}, \delta), H)$, is the smallest integer N such that there exist N brackets of (positive) measures $[P_1^L, P_1^U], \dots, [P_N^L, P_N^U]$ such that $\max_{i \in [N]} H(P_i^L, P_i^U) \leq \varepsilon$, and for every $P \in \mathcal{P}(\bar{P}, \delta)$, there exists $i \in [N]$ such that $P_i^L \leq P \leq P_i^U$ (with \leq being pointwise).*

In the above definition, we extend the definition of Hellinger distance between two probability measures in a natural way to general measures μ, ν by $H^2(\mu, \nu) := \int (\sqrt{d\mu} - \sqrt{d\nu})^2$. The following result states a general entropic upper bound on the Hellinger distance $H(\hat{P}, \bar{P})$.

Lemma 2. *There exists a universal constant $C > 0$ such that the following holds. Let*

$$\Psi(\delta) \geq \int_{\delta^2/C}^{\delta} \sqrt{\log N_{[]}(\nu, \mathcal{P}(\bar{P}, \delta), H)} d\nu \vee \delta$$

be any function such that $\Psi(\delta)/\delta^2$ is non-increasing in δ , and $\delta_n > 0$ satisfy $\sqrt{n}\delta_n^2 \geq C\Psi(\delta_n)$. Then for all $\delta \geq \delta_n$, any β -approximate MLE \hat{P} satisfies

$$\mathbb{P} \left(H^2(\hat{P}, \bar{P}) \geq \delta^2 + \frac{C}{n} \log \frac{1}{\beta} \right) \leq C \exp \left(-\frac{n\delta^2}{C^2} \right).$$

In the i.i.d. case where $P_i \equiv P$, [Lemma 2](#) reduces to the classical upper bounds on $H^2(\hat{P}, P)$ in [[WS95](#), [vdG00](#)]. The same proof technique extends to the i.n.i.d. setting, where the MLE \hat{P} is instead close to the average distribution \bar{P} . In heteroskedastic mean estimation, this average distribution is $\bar{P} = \frac{1}{n} \sum_{i=1}^n \mathcal{N}(\mu, \sigma_i^2) = f_{\mu, G_n}$, which motivates the form of [Theorem 1.3](#). For completeness, we include the proof of [Lemma 2](#) in the appendix, as the i.n.i.d. case in [[vdG00](#), Chapter 8.3] does not cover this precise setting.

2.2 Improved Covering Results for the Normal Scale Mixture

The central metric entropy bound in this section is the following:

Theorem 2.1. *Let $\mathcal{P} = \{f_{\mu, G} : \mu \in \mathbb{R}, \text{supp}(G) \subseteq [\sigma_{\min}, \sigma_{\max}]\}$ and $\bar{P} \in \mathcal{P}$. Then for $0 < \varepsilon \leq \delta \leq \frac{1}{8}$, there is a universal constant $C > 0$ such that*

$$\log N_{[]}(\varepsilon, \mathcal{P}(\bar{P}, \delta), H) \leq C \log^3 \left(\frac{\sigma_{\max}}{\varepsilon \sigma_{\min}} \right) \log^4 \log \left(\frac{\sigma_{\max}}{\varepsilon \sigma_{\min}} \right).$$

In conjunction with [Lemma 2](#), [Theorem 2.1](#) gives the target upper bound of [Theorem 1.3](#). The main technical challenge in the proof of [Theorem 2.1](#) is to derive the bracketing entropy bound for the class of *normal scale mixtures*, i.e., the subset of \mathcal{P} with $\mu = 0$ and a general mixture distribution G on $[\sigma_{\min}, \sigma_{\max}]$. Specifically, using a reparametrization $t = 1/\sigma$ and standard truncation arguments (deferred to [Section C.2](#)), [Theorem 2.1](#) follows from the following technical lemma.

Lemma 3. *Let $0 < t_{\min} \leq 1 \leq t_{\max}$ and $0 < x_{\min} \leq 1 \leq x_{\max}$. Let \mathcal{F} be the class of all functions $\mathbb{E}_{t \sim H}[t \exp(-t^2 x^2/2)]$ with $H \in \mathcal{P}([t_{\min}, t_{\max}])$, and $L_{\infty}([x_{\min}, x_{\max}])$ denotes the sup norm on the interval $[x_{\min}, x_{\max}]$. Then for $\varepsilon \in (0, 1/2)$,*

$$\log N(\varepsilon, \mathcal{F}, L_{\infty}([x_{\min}, x_{\max}])) \leq C \log^3 \left(\frac{t_{\max} x_{\max}}{\varepsilon t_{\min} x_{\min}} \right) \log^4 \log \left(\frac{t_{\max} x_{\max}}{\varepsilon t_{\min} x_{\min}} \right).$$

In [Lemma 3](#), the $\text{polylog}(\frac{1}{\varepsilon})$ dependence on ε is perhaps unsurprising as \mathcal{F} is effectively a parametric family by discretizing the support of H ; the key challenge is to obtain a logarithmic dependence on $\frac{t_{\max}}{t_{\min}}$ and $\frac{x_{\max}}{x_{\min}}$. This is precisely where existing work based on polynomial approximation or Taylor series fail to capture, as detailed in the following sections.

2.2.1 Generalized moment matching

In the literature for normal location or scale mixtures, the prevalent approach for establishing metric entropy bounds is through the idea of *moment matching*. Specifically, it is shown that if H and H' have the same first L moments, then the difference $\|\mathbb{E}_{t \sim H}[t \exp(-t^2 x^2/2)] - \mathbb{E}_{t \sim H'}[t \exp(-t^2 x^2/2)]\|_\infty$ decays exponentially in L . Since Carathéodory's theorem shows that matching first L moments can be realized by a discrete distribution with at most $O(L)$ atoms, covering such discrete distributions typically leads to a metric entropy bound of $\tilde{O}(L)$. This idea has been used for approximating normal scale mixtures in [\[GvdV01, GvdV07, IS25\]](#); an improved bound is also obtained via *local moment matching* in [\[JZ09\]](#) for normal location mixtures, with extensions to Poisson mixtures in [\[SW22\]](#).

However, such moment matching approaches (even the localized version) in normal scale mixtures suffer from a polynomial dependence on the ratio $\frac{t_{\max}}{t_{\min}}$. The underlying mathematical reason is that, such moment matching arguments rely on the following polynomial approximation

$$\sup_{x \in [x_{\min}, x_{\max}], t \in [t_{\min}, t_{\max}]} \left| t e^{-\frac{t^2 x^2}{2}} - \sum_{k=1}^L g_k(x) t^k \right| \leq \varepsilon,$$

and this polynomial is usually chosen to be a truncation of Taylor series. Based on this approximation, it is clear that $\mathbb{E}_{t \sim H}[t \exp(-t^2 x^2/2)]$ approximately depends only on $\mathbb{E}_{t \sim H}[t^k]$, $k = 1, \dots, L$. However, even for a fixed x , approximation theory tells that the degree L must have a polynomial dependence on (t_{\min}, t_{\max}) . For example, [\[AA22\]](#) shows that to approximate e^{-t} with a uniform approximation error ε on $t \in [0, B]$, the degree of the polynomial must be at least $\tilde{\Omega}(\sqrt{B})$, exhibiting a polynomial dependence on B .

To address this issue, we make a simple yet important observation that we may apply a generalized moment matching with suitably chosen basis functions:

$$\sup_{x \in [x_{\min}, x_{\max}], t \in [t_{\min}, t_{\max}]} \left| t e^{-\frac{t^2 x^2}{2}} - \sum_{k=1}^L a_k(t) g_k(x) \right| \leq \varepsilon. \quad (5)$$

Here by choosing $a_k(t)$ beyond polynomials, we may find a better approximating basis of $t \exp(-t^2 x^2/2)$ to achieve a smaller approximation error. The following lemma is an easy metric entropy bound if we can construct bounded functions $\{a_k(t), g_k(x)\}_{k=1}^L$ such that (5) holds.

Lemma 4. *Suppose (5) holds with $|a_k(t)| + t|a'_k(t)| \leq A$ for all $t \in [t_{\min}, t_{\max}]$ and $|g_k(x)| \leq G$ for all $x \in [x_{\min}, x_{\max}]$. Then $\log N(2\varepsilon, \mathcal{F}, L_\infty([x_{\min}, x_{\max}])) \leq CL \log \frac{AGL}{\varepsilon}$.*

To choose the basis, we use the natural idea of writing $u = \log x + \log t$ and try to approximate $\exp(-t^2 x^2/2) = \exp(-\frac{1}{2}e^{2u}) \approx P_L(u)$, where P_L is a polynomial of degree at most L . Expanding $P_L(u)$ into a bivariate polynomial in $(\log x, \log t)$, we obtain the basis functions $(a_k(t), g_k(x))$ in (5) in the form of $t(\log t)^i$ and $(\log x)^j$. Therefore, it remains to find a uniform polynomial approximation for $\exp(-\frac{1}{2}e^{2u})$ on $u \in [\log(x_{\min} t_{\min}), \log(x_{\max} t_{\max})]$, or equivalently, for $\exp(-K e^{\lambda u})$ on $u \in [-1, 1]$ after translation and scaling, with $K = \frac{1}{2} x_{\min} x_{\max} t_{\min} t_{\max} > 0$ and $\lambda = \log \frac{x_{\max} t_{\max}}{x_{\min} t_{\min}}$.

2.2.2 Chebyshev approximation

In this section we solve the polynomial approximation problem of approximating $\exp(-Ke^{\lambda u})$ on $[-1, 1]$. As commonly used in the moment matching literature, a first natural idea is to apply the Taylor approximating polynomial of $\exp(-Ke^{\lambda u})$ at $u = 0$. However, this leads to a poor approximation: after some algebra, we can show that the n -th Taylor coefficient is $a_n = \frac{\lambda^n}{n!} e^{-K} B_n(-K)$, where $B_n(x) = \sum_{k=0}^n S(n, k)x^k$ is the Bell/Touchard polynomial and $S(n, k)$ is the Stirling number of the second kind. Since $|B_n(-K)|$ grows at a speed faster than K^n , we see that $|a_n| \rightarrow 0$ only if $n \gg \lambda K$. However, since K is polynomial in $(x_{\min}, x_{\max}, t_{\min}, t_{\max})$, this means that the degree of the Taylor approximating polynomial must also be a polynomial in $(x_{\min}, x_{\max}, t_{\min}, t_{\max})$. To improve over this dependence, we critically make use of the *Chebyshev approximation* to achieve an approximation error *independent of K* , as summarized in the following lemma.

Lemma 5. *Let $h(x) = \exp(-Ke^{\lambda x})$ on $[-1, 1]$, with $K > 0$ and $\lambda > 0$. Let P_L be the degree- L Chebyshev polynomial of h . Then $\|h - P_L\|_{\infty} \leq \varepsilon$ for $L = O(\log(\frac{1}{\varepsilon}) + \lambda \log(\frac{\lambda}{\varepsilon}))$.*

Proof. The proof bounds the Chebyshev approximation error using the Bernstein ellipse theorem.

Lemma 6 (Bernstein Ellipse Theorem, [Tre19, Theorem 8.2]). *Let h be analytic in the open Bernstein ellipse $E_{\rho} = \{z \in \mathbb{C} : z = \frac{1}{2}(\rho e^{i\theta} + \rho^{-1}e^{-i\theta}), \theta \in [0, 2\pi]\}$ for some $\rho > 1$, and suppose $|h(z)| \leq M$ on E_{ρ} . Let P_L be the degree- L Chebyshev truncation of h on $[-1, 1]$. Then $\|h - P_L\|_{\infty} \leq \frac{2M}{\rho-1}\rho^{-L}$.*

To conclude from Lemma 6, note that the Bernstein ellipse E_{ρ} is contained in the strip $|\Im z| \leq \frac{1}{2}(\rho - \rho^{-1}) \leq \rho - 1$ for $\rho \in [1, 2]$. In addition, $|h(z)| = \exp(-Ke^{\lambda x} \cos(\lambda y))$ for $z = x + iy$. Therefore, for $\rho = \min\{2, 1 + \frac{\pi}{2\lambda}\}$, we have $\cos(\lambda y) \geq 0$ for $|y| \leq \rho - 1$ and thus $|h(z)| \leq 1$ on E_{ρ} . Now Lemma 5 directly follows from Lemma 6. \square

Importantly, the final polynomial degree L in Lemma 5 is only polynomial in λ , and thus polylogarithmic in $\frac{x_{\max}t_{\max}}{x_{\min}t_{\min}}$; this is the target feature of Lemma 3. It is interesting to note why Chebyshev approximation performs better than Taylor approximation here. In fact, Taylor approximation gives a convergence rate ρ^{-n} when $|h|$ is bounded on the disc $|z| = \rho$, whereas Chebyshev approximation operates on a different geometry E_{ρ} which more tightly envelopes the desired interval. Since $|h(z)|$ could grow rapidly along the imaginary axis, here the Bernstein ellipse is much more suitable than the disc.

3 Bounding modulus of continuity

In this section, we establish upper bounds of the Hellinger modulus of continuity in (4), with various choices of G_n . Specifically, we prove the upper bound in Lemma 1 for the subset-of-signals problem, and upper bounds in Corollary 1.1 for explicit examples of G_n .

3.1 Modulus of Continuity for SoS Priors: Proof of Lemma 1

The proof of Lemma 1 relies on the following variational lower bound of squared Hellinger distance:

$$H^2(P, Q) = \int \frac{(P - Q)^2}{(\sqrt{P} + \sqrt{Q})^2} \geq \frac{1}{2} \int \frac{(P - Q)^2}{P + Q} = \frac{1}{2} \sup_T \frac{(\mathbb{E}_P[T] - \mathbb{E}_Q[T])^2}{\mathbb{E}_P[T^2] + \mathbb{E}_Q[T^2]}, \quad (6)$$

where the supremum is over all test functions $T : \mathcal{X} \rightarrow \mathbb{R}$, and the last step is Cauchy–Schwarz. In the definition of Hellinger modulus of continuity, suppose the choices $P = f_{\mu,G}$ and $Q = f_{0,G}$ satisfy $H^2(P, Q) \leq t$. Now choosing test function $T_\Delta(x) = \mathbf{1}_{[\mu, \mu+\Delta]}(x)$, the variational form gives

$$\frac{\left[\mathbb{E}_G \left(\Phi \left(\frac{\Delta}{\sigma} \right) - \Phi(0) - \Phi \left(\frac{\Delta+\mu}{\sigma} \right) + \Phi \left(\frac{\mu}{\sigma} \right) \right) \right]^2}{\mathbb{E}_G \left(\Phi \left(\frac{\Delta}{\sigma} \right) - \Phi(0) + \Phi \left(\frac{\Delta+\mu}{\sigma} \right) - \Phi \left(\frac{\mu}{\sigma} \right) \right)} \leq 2t. \quad (7)$$

Here $\Phi(x) = \int_{-\infty}^x \varphi(t) dt$ denote the standard Gaussian CDF. Taking $\Delta = \infty$, and noting that the denominator always lies in $[0, 1]$, we obtain

$$\sqrt{2t} \geq \mathbb{E}_G \left(\Phi \left(\frac{\mu}{\sigma} \right) - \Phi(0) \right) \geq \Phi'(1) \cdot \mathbb{E}_G \left[\frac{\mu \mathbf{1}_{\{\sigma \geq \mu\}}}{\sigma} \right]$$

due to $\Phi(x) - \Phi(0) \geq \Phi'(1)x$ for $x \in [0, 1]$. This shows that for a universal $c_1 > 0$,

$$\mathbb{E}_G \left[\frac{\mathbf{1}_{\{\sigma \geq \mu\}}}{\sigma} \right] \leq \frac{c_1 \sqrt{t}}{\mu}. \quad (8)$$

Next we take $\Delta = 1$ in (27). For the denominator, we have

$$\begin{aligned} \mathbb{E}_G \left(\Phi \left(\frac{1}{\sigma} \right) - \Phi(0) + \Phi \left(\frac{1+\mu}{\sigma} \right) - \Phi \left(\frac{\mu}{\sigma} \right) \right) &\leq 2 \mathbb{E}_G \left(\Phi \left(\frac{1}{\sigma} \right) - \Phi(0) \right) \\ &\leq 2 \left(\frac{1}{2} G([0, 1]) + \Phi'(0) \cdot \mathbb{E}_G \left[\frac{\mathbf{1}_{\{\sigma \geq 1\}}}{\sigma} \right] \right) \\ &\leq G([0, 1]) + \mathbb{E}_G \left[\frac{\mathbf{1}_{\{\sigma \geq 1\}}}{\sigma} \right]. \end{aligned} \quad (9)$$

For the numerator, we distinguish into two cases.

Case I: $\mu \geq 1$. Let

$$\Xi_{\mu, \sigma} := \Phi \left(\frac{1}{\sigma} \right) - \Phi(0) - \Phi \left(\frac{1+\mu}{\sigma} \right) + \Phi \left(\frac{\mu}{\sigma} \right).$$

When $\mu \geq 1$, simple algebra yields that for a universal constant $c_2 > 0$,

$$\Xi_{\mu, \sigma} \geq c_2 \begin{cases} 1 & \text{if } \sigma < 1, \\ \frac{1}{\sigma} & \text{if } 1 \leq \sigma < \mu, \\ 0 & \text{if } \sigma \geq \mu. \end{cases} \quad (10)$$

Therefore,

$$\mathbb{E}_G[\Xi_{\mu, \sigma}] \geq c_2 \left(G([0, 1]) + \mathbb{E}_G \left[\frac{\mathbf{1}_{\{1 \leq \sigma \leq \mu\}}}{\sigma} \right] \right), \quad (11)$$

and

$$\begin{aligned} \left(G([0, 1]) + \mathbb{E}_G \left[\frac{\mathbf{1}_{\{1 \leq \sigma \leq \mu\}}}{\sigma} \right] \right)^2 &\stackrel{(a)}{\leq} c_3 t \left(G([0, 1]) + \mathbb{E}_G \left[\frac{\mathbf{1}_{\{\sigma \geq 1\}}}{\sigma} \right] \right) \\ &\stackrel{(b)}{\leq} c_3 t \left(G([0, 1]) + \mathbb{E}_G \left[\frac{\mathbf{1}_{\{1 \leq \sigma \leq \mu\}}}{\sigma} \right] + \frac{c_1 \sqrt{t}}{\mu} \right). \end{aligned}$$

Here (a) follows from (27) and (29), and (b) uses (28). Solving this quadratic inequality gives

$$G([0, 1]) + \mathbb{E}_G \left[\frac{\mathbf{1}_{\{1 \leq \sigma \leq \mu\}}}{\sigma} \right] \leq c_4 \left(t + \frac{t^{3/4}}{\mu^{1/2}} \right).$$

As $G([0, 1]) \geq p \geq Ct$ for $C \geq 2c_4$, this inequality gives that $\mu = O\left(\frac{t^{3/2}}{p^2}\right)$.

Case II: $0 \leq \mu < 1$. In this case, one can verify the following lower bounds for $\Xi_{\mu,\sigma}$:

$$\Xi_{\mu,\sigma} \geq c_2 \begin{cases} 1 & \text{if } \sigma < \mu, \\ \frac{\mu}{\sigma} & \text{if } \mu \leq \sigma < 1, \\ 0 & \text{if } \sigma \geq 1. \end{cases} \quad (12)$$

Therefore, the numerator is lower bounded as

$$\mathbb{E}_G[\Xi_{\mu,\sigma}] \geq c_2 \left(G([0, \mu]) + \mathbb{E}_G \left[\frac{\mu \mathbf{1}_{\{\mu \leq \sigma \leq 1\}}}{\sigma} \right] \right) \geq c_2 \mu \cdot G([0, 1]). \quad (13)$$

Consequently, by (27), (28), and (29), we get

$$\mu^2 G([0, 1])^2 \leq c_3 t \left(G([0, 1]) + \mathbb{E}_G \left[\frac{\mathbf{1}_{\{\sigma \geq 1\}}}{\sigma} \right] \right) \leq c_3 t \left(G([0, 1]) + \frac{c_1 \sqrt{t}}{\mu} \right),$$

and further solve this quadratic inequality to find

$$G([0, 1]) \leq c_4 \left(\frac{t}{\mu^2} + \frac{t^{3/4}}{\mu^{3/2}} \right).$$

Finally, as $G([0, 1]) \geq p$, we obtain $\mu = O\left(\frac{t^{1/2}}{p^{1/2}} + \frac{t^{1/2}}{p^{2/3}}\right) = O\left(\frac{t^{1/2}}{p^{2/3}}\right)$ as $p \leq 1$.

To reach the desired conclusion, note that when $p^{4/3} < t \leq \frac{p}{C}$, we have either $\mu = O\left(\frac{t^{3/2}}{p^2}\right)$ (Case I), or $\mu \leq 1$ (assumption of Case II). Since the upper bound in Case I is larger, we conclude that $\mu = O\left(\frac{t^{3/2}}{p^2}\right)$. Similarly, when $t \leq p^{4/3}$, we have either $\mu = O\left(\frac{t^{3/2}}{p^2}\right)$ (Case I), or $\mu = O\left(\frac{t^{1/2}}{p^{2/3}}\right)$ (Case II). Here the upper bound in Case II dominates, so that $\mu = O\left(\frac{t^{1/2}}{p^{2/3}}\right)$.

3.2 Modulus of Continuity for Selected Priors: Proof of Corollary 1.1

To prove Corollary 1.1, we invoke Theorem 1.1 and establish upper bounds of the Hellinger modulus of continuity for various choices of G_n .

Equal variance. When $\sigma_i \equiv 1$, we have $G_n = \delta_1$. In this case,

$$H^2(f_{\mu_1, G_n}, f_{\mu_2, G_n}) = 2 - 2 \exp\left(-\frac{(\mu_1 - \mu_2)^2}{8}\right),$$

so that $\omega_{H^2, G_n}(t) = O(\sqrt{t})$. Now plugging in $t = \tilde{O}\left(\frac{1}{n}\right)$ gives the result.

Quadratic variance. Consider any $\mu > 0$ such that (27) holds for all $\Delta \geq 0$, with $G = G_n$. Next we choose $\Delta = \mu$ in (27) and proceed in the same way as (29) to obtain

$$G_n([0, \mu])^2 \leq c_1 t \left(G_n([0, \mu]) + \mu \mathbb{E}_{G_n} \left[\frac{\mathbf{1}_{\{\sigma \geq \mu\}}}{\sigma} \right] \right).$$

For $t \leq c_2$ with a small enough constant $c_2 > 0$, the above inequality shows that $\mu > n$ is impossible. For $1 \leq \mu \leq n$, since $\mathbb{E}_{G_n} \left[\frac{\mathbf{1}_{\{\sigma \geq \mu\}}}{\sigma} \right] \leq \mathbb{E}_{G_n} \left[\frac{1}{\sigma} \right] = \frac{1}{n} \sum_{i=1}^n \frac{1}{i} = O\left(\frac{\log n}{n}\right)$ and $G_n([0, \mu]) \asymp \frac{\mu}{n}$, we can solve that $\mu = O(nt \log n)$. Combining with the remaining case $\mu \leq 1$ leads to the final upper bound

$$\omega_{H^2, G_n}(t) = O(nt \log n \vee 1), \quad \text{if } t \leq c_2.$$

Plugging in $t = \tilde{O}\left(\frac{1}{n}\right)$ proves the target result.

Two variances. Again, consider some $\mu > 0$ such that (27) holds for every $\Delta \geq 0$, with $G = G_n = \frac{m}{n}\delta_1 + (1 - \frac{m}{n})\delta_\sigma$. We consider two choices of Δ . First, choosing $\Delta = \sigma$ and upper bounding the denominator by 2 in (27), we get

$$\frac{n-m}{n} \left(\Phi(1) - \Phi(0) - \Phi(1 + \frac{\mu}{\sigma}) + \Phi(\frac{\mu}{\sigma}) \right) \leq c_1 \sqrt{t}.$$

Clearly, for $t \leq c_2$ with a small enough constant $c_2 > 0$, this inequality implies that $\mu \leq \sigma$. In this case, the left-hand side further scales as $\Omega(\frac{\mu}{\sigma})$, so that we get

$$\omega_{H^2, G_n}(t) = O(\sigma \sqrt{t}), \quad \text{if } t \leq c_2.$$

For $t = \tilde{O}(\frac{1}{n})$, this gives the general $O(\frac{\sigma}{\sqrt{n}})$ upper bound.

The second choice of Δ is $\Delta = 1$. If we upper bound the denominator of (27) by 2, then

$$\frac{m}{n} (\Phi(1) - \Phi(0) - \Phi(1 + \mu) + \Phi(\mu)) \leq c_1 \sqrt{t}.$$

Therefore, if $\frac{n\sqrt{t}}{m} \leq c_2$ for a small enough constant $c_2 > 0$, we solve that $\mu = O(\frac{n\sqrt{t}}{m})$. For $t = \tilde{O}(\frac{1}{n})$, this proves the $\tilde{O}(\frac{\sqrt{n}}{m})$ upper bound for $m \geq \sqrt{n \log n}$. Instead, if we upper bound the denominator of (27) via (29), we obtain

$$\frac{m}{n} (\Phi(1) - \Phi(0) - \Phi(1 + \mu) + \Phi(\mu)) \leq c_1 \sqrt{t \left(\frac{m}{n} + \frac{1}{\sigma} \right)}.$$

For $t = \tilde{O}(\frac{1}{n})$ and $m \geq C(\sqrt{\frac{n \log n}{\sigma}} + \log n)$, it holds that

$$\frac{m}{n} \geq 10c_1 \sqrt{t \left(\frac{m}{n} + \frac{1}{\sigma} \right)}.$$

Solving the inequality gives the target upper bound $\mu = O(1)$.

α -mixture distributions. We use the upper bound in the two variances example. For $\alpha \in (0, 1)$, we use the upper bound $\tilde{O}(\frac{\sigma}{\sqrt{n}}) = \tilde{O}(n^{\alpha-1/2})$. For $\alpha \geq 1$, we note that for $m = c \log n$ with a large enough constant $c > 0$, it holds that

$$C \left(\sqrt{\frac{n \log n}{n^\alpha}} + \log n \right) \leq m < \sqrt{n \log n}.$$

Therefore, we can apply the $O(1)$ upper bound in this case.

4 Numerical experiments and algorithms

In this section, we summarize important properties of the NPMLE that are well-known in the empirical Bayes literature [Lin83, Lin95], specialized to normal scale mixtures. We also detail our computational approach for finding the NPMLE based on the Frank–Wolfe procedure adapted from [HNWSW25], and numerically compare our approach to previous methods in the literature such as median and iterative truncation [LY20, PJJ22].

4.1 Basic properties of the NPMLE

The main difficulty in solving the MLE (3) is to solve for the NPMLE \hat{G} once $\hat{\mu}$ is fixed. This section describes some basic properties of \hat{G} , and WLOG we assume that $\hat{\mu} = 0$. Since the log-likelihood is concave in G , it is well-known [Lin83] that the KKT condition for \hat{G} is

$$D_{\hat{G}}(\sigma) := D_{\hat{G}}(\sigma; X^n) := \frac{1}{n} \sum_{i=1}^n \frac{\frac{1}{\sigma} \varphi\left(\frac{X_i}{\sigma}\right)}{f_{0, \hat{G}}(X_i)} \leq 1, \quad \forall \sigma \geq 0, \quad (14)$$

and equality holds if and only if $\sigma \in \text{supp}(\hat{G})$. Based on this, we have the following structural result.

Lemma 7. *Suppose $X_i \neq 0$ for all $i \in [n]$. Then the NPMLE \hat{G} is a discrete distribution with $|\text{supp}(\hat{G})| \leq \{|\{X_1\}, \dots, \{X_n\}|\}$, $\text{supp}(\hat{G}) \subseteq [\min_{i \in [n]} |X_i|, \max_{i \in [n]} |X_i|]$, and unique.*

4.2 Successive maximization algorithm for the joint MLE

Motivated by the NPMLE properties in Lemma 7, especially the discrete nature of \hat{G} , we use a fully-corrective Frank–Wolfe algorithm [FW56] (also known as the vertex-direction method in the mixture model literature) to compute \hat{G} with fixed $\hat{\mu}$, in a similar manner to [HNWSW25]. Specifically, given an initialization \hat{G}_0 , for $t = 1, 2, \dots$, we repeat the following steps until convergence: (1) find a new atom $\sigma_t \in \arg \max_{\sigma > 0} D_{\hat{G}_{t-1}}(\sigma; X^n - \hat{\mu})$ via a grid search, and then (2) add σ_t to the support of \hat{G} and optimize over the weights via a convex program:

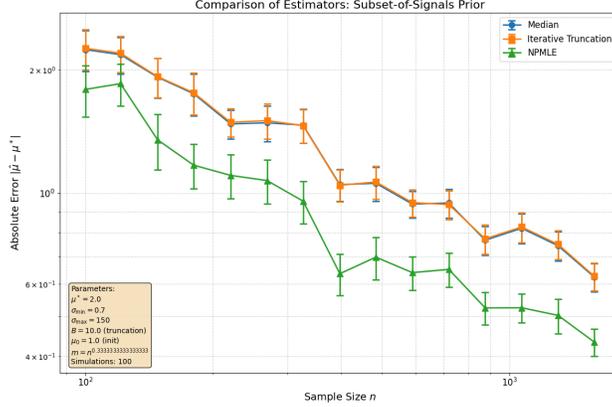
$$\hat{G}_t = \arg \max_{\text{supp}(G) \subseteq \text{supp}(\hat{G}_{t-1}) \cup \{\sigma_t\}} \frac{1}{n} \sum_{i=1}^n \log f_{\hat{\mu}, G}(X_i).$$

We note that in our implementation we do not enforce the constraints $\text{supp}(\hat{G}) \subseteq [\sigma_{\min}, \sigma_{\max}]$; this is because by Lemma 7, the data X^n have already constrained the support of \hat{G} . Based on the Frank–Wolfe algorithm for fixed $\hat{\mu}$, our final algorithm for joint optimization of $(\hat{\mu}, \hat{G})$ proceeds via successive maximization. Given an initialization $\hat{\mu}_0$ (chosen as the median in our implementation), for $t = 1, 2, \dots$, we iterate the following steps until convergence: (1) run the Frank–Wolfe algorithm with $\hat{\mu}$ fixed at $\hat{\mu}_{t-1}$ to obtain \hat{G}_t ; (2) perform a grid search to find $\hat{\mu}_t$ that maximizes the total likelihood for fixed \hat{G}_t . Since the likelihood improves at each iteration, we terminate when the increase in likelihood falls below a prescribed threshold. Although the overall algorithm is susceptible to local maxima, our experiments show that it converges well in practice.

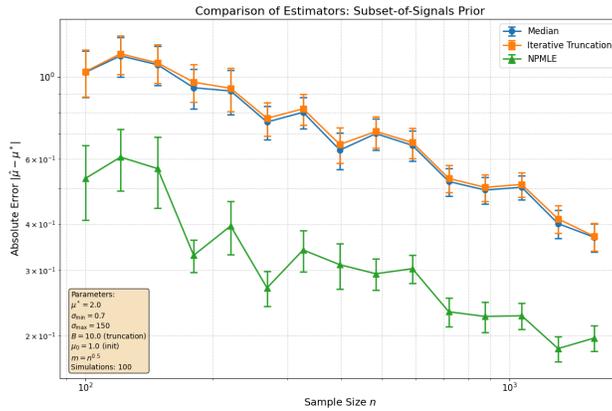
4.3 Experiments

In this section, we illustrate the empirical performance of our estimator $\hat{\mu}^{\text{EB}}$ on a variety of empirical distributions (priors) G_n of $(\sigma_1, \dots, \sigma_n)$, including the subset-of-signals prior $G_n = \frac{m}{n} \text{Unif}([\sigma_{\min}, 1]) + \frac{n-m}{n} \text{Unif}([1, \sigma_{\max}])$, as well as two-point and three-point scale mixture priors. We conduct two sets of experiments. On the subset-of-signals prior, we compare our estimator with two other estimators designed for the subset-of-signal problem: the sample median, and the iterative truncation estimator in [LY20]. On the two-point and three-point priors, in addition to the median estimator, we compare with two *oracle* estimators: the linear estimator $\hat{\mu}_{\text{known-}\sigma}$ in (1) which we call the oracle MLE, and the MLE of μ in (3) with the true knowledge of G_n . We emphasize that apart from the grid search and stopping criterion, our algorithm is completely free of tuning parameters.

Figure 1 and 2 display the average absolute estimation error of μ over 100 simulations for both sets of experiments. In Figure 1, we set $(\sigma_{\min}, \sigma_{\max}) = (0.7, 150)$ and $m \in \{n^{1/3}, n^{1/2}\}$, and the



(a) $m = n^{1/3}$



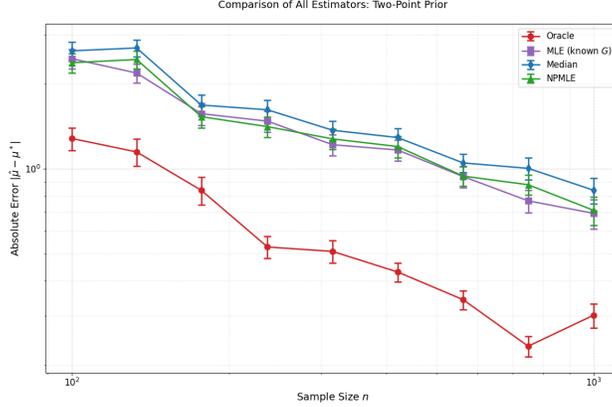
(b) $m = \sqrt{n}$

Figure 1: Average absolute estimation errors for $\hat{\mu}^{\text{EB}}$, sample median, and iterative truncation over 100 simulations, under the subset-of-signals prior $G_n = \frac{m}{n} \text{Unif}([0.7, 1]) + \frac{n-m}{n} \text{Unif}([1, 150])$, with different choices of (m, n) .

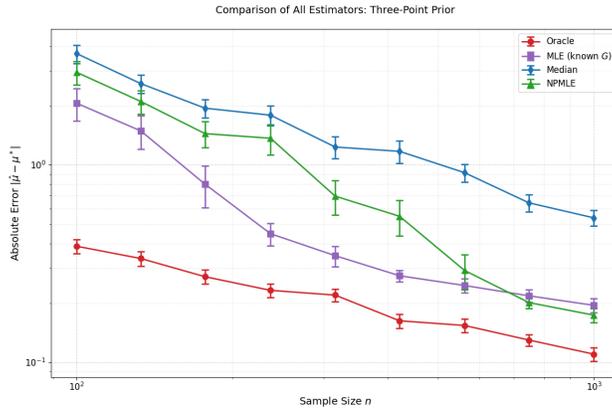
hyperparameters $(B, \mu_0) = (10, 1)$ for the iterative truncation estimator. We observe that $\hat{\mu}^{\text{EB}}$ outperforms the other two estimators in both settings. In Figure 2, the scale mixture is a two-point prior $G_n = \frac{1}{100} \delta_1 + \frac{99}{100} \delta_{30}$ or three-point prior $G_n = \frac{1}{20} \delta_1 + \frac{9}{10} \delta_{60} + \frac{1}{20} \delta_{100}$. We observe that the performance of $\hat{\mu}^{\text{EB}}$, obtained without knowledge of G_n , is close to that of the MLE computed with knowledge of G_n . This indicates that the proposed successive maximization algorithm, in particular the Frank–Wolfe procedure used to estimate \hat{G} , exhibits good empirical convergence, despite the potential for convergence to local minima.

5 Conclusion and open problems

In this work, we demonstrate that a simple empirical Bayes approach based on maximum likelihood estimation achieves near-optimal performance for heteroskedastic mean estimation across a wide range of instances. Compared with traditional empirical Bayes approaches, our procedure replaces heterogeneity with a mixing distribution (prior) and learns it from the data, but does not invoke explicit posterior inference. Instead, the learned prior is treated as a nuisance estimate and is used in constructing our final mean estimator, which takes the form of a profile MLE. It is thus an interesting



$$(a) G_n = \frac{1}{100}\delta_1 + \frac{99}{100}\delta_{30}$$



$$(b) G_n = \frac{1}{20}\delta_1 + \frac{9}{10}\delta_{60} + \frac{1}{20}\delta_{100}$$

Figure 2: Average absolute estimation errors for $\hat{\mu}^{\text{EB}}$, sample median, and two oracle MLEs over 100 simulations, under two-point and three-point scale mixture priors.

finding that this empirical Bayes approach still achieves near-optimal statistical performance in heteroskedastic mean estimation. Several interesting questions remain open.

Efficient computation. From a computational perspective, the optimization problem in (3) is nonconvex (in μ) and infinite-dimensional (in G). While our numerical experiments suggest that successive maximization with Frank–Wolfe performs well in practice, it would be valuable to develop provably efficient algorithms and/or to understand the landscape of this optimization problem.

Hyperparameters ($\sigma_{\min}, \sigma_{\max}$). Our theoretical guarantee involves a logarithmic factor in $\frac{\sigma_{\max}}{\sigma_{\min}}$, which primarily arises from our proof strategy for establishing the density estimation guarantee in Theorem 1.3. Indeed, when $\sigma_{\min} = 0$, the likelihood in (3) can be made unbounded by assigning positive mass to $\hat{G}(0)$ and choosing $\hat{\mu} \in \{X_1, \dots, X_n\}$, in which case $H^2(f_{\hat{\mu}, \hat{G}}, f_{\mu, G})$ is no longer small. However, this does *not* imply that the resulting estimator $\hat{\mu}$ fails to estimate μ accurately. For example, in our numerical experiments, we always set $\sigma_{\min} = 0$ and $\sigma_{\max} = \infty$, yet the resulting estimator continues to perform well. As another illustration, the Le Cam–Birgé-type mean estimator studied in [CV25] likewise satisfies $\hat{\mu} \in \{X_1, \dots, X_n\}$, suggesting that estimators of this form can still achieve good performance. It is therefore an interesting question to remove the assumptions on $(\sigma_{\min}, \sigma_{\max})$.

Le Cam’s lower bound for compound problems. Although $\omega_{H^2, G_n}(\frac{1}{n})$ is a minimax lower bound for the mean estimation problem where X_1, \dots, X_n are i.i.d. drawn from f_{μ, G_n} with known G_n , the same claim is not rigorous in the compound setting. In compound setting, Le Cam’s method only gives the following lower bound: let $\mathbb{P}_{\mu, G_n} := \mathbb{E}_{\pi \sim \text{Unif}(S_n)}[\bigotimes_{i=1}^n \mathcal{N}(\mu, \sigma_{\pi(i)}^2)]$ be the n -dimensional “permutation mixture”, then a minimax lower bound for mean estimation is

$$\sup \left\{ |\mu_1 - \mu_2| : \mu_1, \mu_2 \in \mathbb{R}, H^2(\mathbb{P}_{\mu_1, G_n}, \mathbb{P}_{\mu_2, G_n}) \leq 2 - \Omega(1) \right\}.$$

Assuming a mean-field approximation $\mathbb{P}_{\mu, G_n} \approx f_{\mu, G_n}^{\otimes n}$ for permutation mixtures, the above quantity would essentially reduce to $\omega_{H^2, G_n}(\frac{1}{n})$. However, although recent work [HNW24, LH25] show that such a mean-field approximation holds quantitatively and independently of the dimension n , these results only guarantee $\text{TV}(\mathbb{P}_{\mu, G_n}, f_{\mu, G_n}^{\otimes n}) \leq 1 - c$ for a small constant $c > 0$. Consequently, a direct application of the triangle inequality yields a vacuous bound (incurring a cost of $2 \cdot \text{TV}$) and is therefore insufficient to establish the desired equivalence.

Multivariate settings. Extending our framework to multivariate settings, where observations $X_i \in \mathbb{R}^d$ have heteroskedastic covariance matrices, presents both statistical and computational challenges. On the statistical side, the density estimation-based proof strategy becomes unsuitable, since the Hellinger error typically scales exponentially in d . On the computational side, computing the NPMLE in high dimensions is challenging, and standard algorithms such as Frank–Wolfe become less effective.

References

- [AA22] Amol Aggarwal and Josh Alman. Optimal-degree polynomial approximations for exponentials and gaussian kernel density estimation. *arXiv preprint arXiv:2205.06249*, 2022.
- [BG09] Lawrence D Brown and Eitan Greenshtein. Nonparametric empirical bayes and compound decision approaches to estimation of a high-dimensional vector of normal means. *The Annals of Statistics*, pages 1685–1704, 2009.
- [BGR13] Lawrence D Brown, Eitan Greenshtein, and Ya’acov Ritov. The poisson compound decision problem revisited. *Journal of the American Statistical Association*, 108(502):741–749, 2013.
- [Bic65] Peter J Bickel. On some robust estimates of location. *The Annals of Mathematical Statistics*, 36(3):847–858, 1965.
- [BM98] Lucien Birgé and Pascal Massart. Minimum contrast estimators on sieves: exponential bounds and rates of convergence. *Bernoulli*, 4(3):329–375, 1998.
- [CDKL14] Flavio Chierichetti, Anirban Dasgupta, Ravi Kumar, and Silvio Lattanzi. Learning entangled single-sample gaussians. In *Proceedings of the twenty-fifth annual ACM-SIAM symposium on Discrete algorithms*, pages 511–522. SIAM, 2014.
- [CGR18] Mengjie Chen, Chao Gao, and Zhao Ren. Robust covariance and scatter matrix estimation under Huber’s contamination model. In *The Annals of Statistics*, volume 46, pages 1932–1960, 2018.

- [CV24] Spencer Compton and Gregory Valiant. Near-optimal mean estimation with unknown, heteroskedastic variances. In *Proceedings of the 56th Annual ACM Symposium on Theory of Computing*, pages 194–200, 2024.
- [CV25] Spencer Compton and Gregory Valiant. Attainability of two-point testing rates for finite-sample location estimation. *arXiv preprint arXiv:2502.05730*, 2025.
- [DKK⁺19] Ilias Diakonikolas, Gautam Kamath, Daniel Kane, Jerry Li, Ankur Moitra, and Alstair Stewart. Robust estimators in high-dimensions without the computational intractability. In *SIAM Journal on Computing*, volume 48, pages 742–864. SIAM, 2019.
- [DKLP25] Ilias Diakonikolas, Daniel M Kane, Sihan Liu, and Thanasis Pittas. Entangled mean estimation in high dimensions. In *Proceedings of the 57th Annual ACM Symposium on Theory of Computing*, pages 1680–1688, 2025.
- [DL91] David L Donoho and Richard C Liu. Geometrizing rates of convergence, II. *The Annals of Statistics*, 19(2):633–667, 1991.
- [DLLZ23] Luc Devroye, Silvio Lattanzi, Gábor Lugosi, and Nikita Zhivotovskiy. On mean estimation for heteroscedastic random variables. In *Annales de l’Institut Henri Poincaré (B) Probabilités et statistiques*, volume 59, pages 1–20. Institut Henri Poincaré, 2023.
- [DLR77] Arthur P Dempster, Nan M Laird, and Donald B Rubin. Maximum likelihood from incomplete data via the EM algorithm. *Journal of the Royal Statistical Society: Series B (Methodological)*, 39(1):1–38, 1977.
- [Efr24] Bradley Efron. Empirical bayes: Concepts and methods. In *Handbook of Bayesian, Fiducial, and Frequentist Inference*, pages 8–34. Chapman and Hall/CRC, 2024.
- [ETST01] Bradley Efron, Robert Tibshirani, John D Storey, and Virginia Tusher. Empirical bayes analysis of a microarray experiment. *Journal of the American Statistical Association*, 96(456):1151–1160, 2001.
- [FW56] Marguerite Frank and Philip Wolfe. An algorithm for quadratic programming. *Naval Research Logistics Quarterly*, 3(1-2):95–110, 1956.
- [GIKL25] Sulagna Ghosh, Nikolaos Ignatiadis, Frederic Koehler, and Amber Lee. Stein’s unbiased risk estimate and Hyvärinen’s score matching. *arXiv preprint arXiv:2502.20123*, 2025.
- [GvdV01] Subhashis Ghosal and Aad W van der Vaart. Entropies and rates of convergence for maximum likelihood and Bayes estimation for mixtures of normal densities. *The Annals of Statistics*, 29(5):1233–1263, 2001.
- [GvdV07] S Ghosal and AW van der Vaart. Posterior convergence rates of dirichlet mixtures at smooth densities. *Annals of Statistics*, 35(2):697–723, 2007.
- [GW00] Christopher R Genovese and Larry Wasserman. Rates of convergence for the Gaussian mixture sieve. *The Annals of Statistics*, 28(4):1105–1127, 2000.
- [Ham71] Frank R Hampel. A general qualitative definition of robustness. *The Annals of Mathematical Statistics*, 42(6):1887–1896, 1971.

- [HNW24] Yanjun Han and Jonathan Niles-Weed. Approximate independence of permutation mixtures. *arXiv preprint arXiv:2408.09341*, 2024.
- [HNWSW25] Yanjun Han, Jonathan Niles-Weed, Yandi Shen, and Yihong Wu. Besting good-turing: Optimality of non-parametric maximum likelihood for distribution estimation. *arXiv preprint arXiv:2509.07355*, 2025.
- [HRRS86] Frank R Hampel, Elvezio M Ronchetti, Peter J Rousseeuw, and Werner A Stahel. *Robust Statistics: The Approach Based on Influence Functions*. John Wiley & Sons, 1986.
- [Hub64] Peter J Huber. Robust estimation of a location parameter. *The Annals of Mathematical Statistics*, 35(1):73–101, 1964.
- [Hub81] Peter J Huber. *Robust Statistics*. John Wiley & Sons, 1981.
- [IS25] Nikolaos Ignatiadis and Bodhisattva Sen. Empirical Bayes. *Lecture notes*, 2025.
- [Jag13] Martin Jaggi. Revisiting Frank-Wolfe: Projection-free sparse convex optimization. In *International Conference on Machine Learning*, pages 427–435. PMLR, 2013.
- [JN09] Anatoli B Juditsky and Arkadi S Nemirovski. Nonparametric estimation via convex programming. *Annals of statistics*, 37(5):2278–2300, 2009.
- [JPTW23] Soham Jana, Yury Polyanskiy, Anzo Z Teh, and Yihong Wu. Empirical bayes via erm and rademacher complexities: the poisson model. In *The Thirty Sixth Annual Conference on Learning Theory*, pages 5199–5235. PMLR, 2023.
- [JPW25] Soham Jana, Yury Polyanskiy, and Yihong Wu. Optimal empirical bayes estimation for the poisson model via minimum-distance methods. *Information and Inference: A Journal of the IMA*, 14(4):iaaf027, 2025.
- [JZ09] Wenhua Jiang and Cun-Hui Zhang. General maximum likelihood empirical Bayes estimation of normal means. *The Annals of Statistics*, 37(4):1647–1684, 2009.
- [KG17] Roger Koenker and Jiaying Gu. REBayes: an R package for empirical Bayes mixture methods. *Journal of Statistical Software*, 82(8):1–26, 2017.
- [KM14] Roger Koenker and Ivan Mizera. Convex optimization, shape constraints, compound decisions, and empirical Bayes rules. *Journal of the American Statistical Association*, 109(506):674–685, 2014.
- [KW56] Jack Kiefer and Jacob Wolfowitz. Consistency of the maximum likelihood estimator in the presence of infinitely many incidental parameters. *The Annals of Mathematical Statistics*, 27(4):887–906, 1956.
- [Lai78] Nan Laird. Nonparametric maximum likelihood estimation of a mixing distribution. *Journal of the American Statistical Association*, 73(364):805–811, 1978.
- [LC73] Lucien Le Cam. Convergence of estimates under dimensionality restrictions. *The Annals of Statistics*, 1(1):38–53, 1973.
- [LC86] Lucien Le Cam. *Asymptotic Methods in Statistical Decision Theory*. Springer-Verlag, 1986.

- [LH25] Yiguo Liang and Yanjun Han. Sharp mean-field analysis of permutation mixtures and permutation-invariant decisions. *arXiv preprint arXiv:2509.12584*, 2025.
- [Lin83] Bruce G Lindsay. The geometry of mixture likelihoods: a general theory. *The Annals of Statistics*, 11(1):86–94, 1983.
- [Lin95] Bruce G Lindsay. Mixture models: theory, geometry, and applications. Ims, 1995.
- [LK92] Mary L Lesperance and John D Kalbfleisch. An algorithm for computing the nonparametric MLE of a mixing distribution. *Journal of the American Statistical Association*, 87(417):120–126, 1992.
- [LRV16] Kevin A Lai, Anup B Rao, and Santosh Vempala. Agnostic estimation of mean and covariance. In *2016 IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 665–674. IEEE, 2016.
- [LY20] Yingyu Liang and Hui Yuan. Learning entangled single-sample gaussians in the subset-of-signals model. In *Conference on Learning Theory*, pages 2712–2737. PMLR, 2020.
- [MVdV97] Susan A Murphy and Aad W Van der Vaart. Semiparametric likelihood ratio inference. *The Annals of Statistics*, 25(4):1471–1509, 1997.
- [MVdV00] Susan A Murphy and Aad W Van der Vaart. On profile likelihood. *Journal of the American Statistical Association*, 95(450):449–465, 2000.
- [MVdVRS99] Susan A Murphy, Aad W Van der Vaart, James Robins, and Joke PJ Slaets. Current status data with two competing risks. *The Annals of Statistics*, pages 1751–1770, 1999.
- [Nar25] Shyam Narayanan. Personal communication. 2025.
- [PJJ22] Ankit Pensia, Varun Jog, and Po-Ling Loh. Estimating location parameters in sample-heterogeneous distributions. *Information and Inference: A Journal of the IMA*, 11(3):959–1036, 2022.
- [PS25] George Pólya and Gabor Szegő. Aufgaben und lehrsätze aus der analysis: Zweiter band: Funktionentheorie· nullstellen polynome· determinanten zahlentheorie. 1925.
- [PW19] Yury Polyanskiy and Yihong Wu. Dualizing le cam’s method for functional estimation, with applications to estimating the unseens. *arXiv preprint arXiv:1902.05616*, 2019.
- [PW20] Yury Polyanskiy and Yihong Wu. Self-regularizing property of nonparametric maximum likelihood estimator in mixture models. *arXiv preprint arXiv:2008.08244*, 2020.
- [PW21] Yury Polyanskiy and Yihong Wu. Sharp regret bounds for empirical bayes and compound decision problems. *arXiv preprint arXiv:2109.03943*, 2021.
- [Rob51] Herbert Robbins. Asymptotically subminimax solutions of compound statistical decision problems. *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, pages 131–148, 1951.
- [Rob56] Herbert Robbins. An empirical Bayes approach to statistics. *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, 1:157–163, 1956.

- [Sev00] Thomas A Severini. Likelihood methods in statistics. 2000.
- [SG20] Sujayam Saha and Adityanand Guntuboyina. On the nonparametric maximum likelihood estimator for gaussian location mixture densities with application to gaussian denoising. *The Annals of Statistics*, 48(2):738–762, 2020.
- [She97] Xiaotong Shen. Methods of sieves and penalization. *The Annals of Statistics*, 25(6):2555–2591, 1997.
- [Sti73] Stephen M Stigler. Simon Newcomb, Percy Daniell, and the history of robust estimation 1885–1920. *Journal of the American Statistical Association*, 68(344):872–879, 1973.
- [SW22] Yandi Shen and Yihong Wu. Empirical bayes estimation: When does g -modeling beat f -modeling in theory (and in practice)? *arXiv preprint arXiv:2211.12692*, 2022.
- [Tre19] Lloyd N Trefethen. *Approximation theory and approximation practice, extended edition*. SIAM, 2019.
- [Tuk60] John W Tukey. A survey of sampling from contaminated distributions. *Contributions to Probability and Statistics*, pages 448–485, 1960.
- [Tuk75] John W Tukey. Mathematics and the picturing of data. *Proceedings of the International Congress of Mathematicians*, 2:523–531, 1975.
- [vdG00] Sara van de Geer. *Empirical Processes in M-estimation*, volume 6. Cambridge university press, 2000.
- [VdV98] Aad W Van der Vaart. *Asymptotic Statistics*. Cambridge University Press, 1998.
- [WS95] Wing Hung Wong and Xiaotong Shen. Probability inequalities for likelihood ratios and convergence rates of sieve mles. *The Annals of Statistics*, pages 339–362, 1995.
- [Zha03] Cun-Hui Zhang. Compound decision theory and empirical Bayes methods. *The Annals of Statistics*, 31(2):379–390, 2003.
- [Zha09] Cun-Hui Zhang. Generalized maximum likelihood estimation of normal mixture densities. *Statistica Sinica*, pages 1297–1318, 2009.

A Further Related work

Heteroskedastic mean estimation. The problem of estimating a common location parameter from heterogeneous observations has attracted significant recent attention in both the statistics and theoretical computer science communities. Pensia, Jog, and Loh [P JL22] initiated the systematic study of this problem, characterizing minimax rates for location estimation under sample-heterogeneous distributions and proposing estimators based on iterative trimming and median-of-means techniques. Devroye, Lattanzi, Lugosi, and Zhivotovskiy [DLLZ23] further developed this line of work, obtaining sharp minimax bounds for heteroscedastic mean estimation under bounded variance assumptions and establishing the phase transition at $m \asymp n^{1/4}$ in the subset-of-signals regime. Both works reveal a fundamental tension in this setting: the sample mean is efficient when

variances are known but highly sensitive to large variances, while robust alternatives such as the median [Hub64, HRRS86] are resilient but often suboptimal.

The heteroscedastic setting studied here differs from the classical Huber contamination model [Hub64, Hub81], where an ε -fraction of observations are adversarially corrupted. In the contamination model, robust estimators aim for error scaling with ε [DKK⁺19, LRV16, CGR18], whereas in our setting the noise levels can vary continuously across samples, leading to different minimax rates and requiring distinct estimation strategies.

Robust statistics and breakdown point. Classical robust statistics provides a rich toolkit for handling outliers and heavy-tailed distributions. Foundational work by Tukey [Tuk60, Tuk75], Huber [Hub64, Hub81], and Hampel [Ham71, HRRS86] established the concepts of breakdown point, influence function, and M-estimation that underpin modern robust methods. The median achieves the optimal breakdown point of $1/2$, meaning it remains bounded even when nearly half the observations are arbitrarily corrupted. Trimmed means [Bic65, Sti73] offer a tunable tradeoff between efficiency and robustness. However, these classical estimators do not achieve the optimal rates in the heteroscedastic setting characterized by [PJL22, DLLZ23], motivating the search for new approaches.

Empirical Bayes and compound decision theory. Empirical Bayes methodology, pioneered by Robbins [Rob51, Rob56], provides a principled framework for estimation problems with many latent parameters. In the compound decision setting, one observes $X_i \sim P_{\theta_i}$ for $i = 1, \dots, n$ and aims to estimate the parameter vector $(\theta_1, \dots, \theta_n)$. The empirical Bayes approach treats the parameters as draws from an unknown prior G and estimates G from the marginal distribution of the observations. The nonparametric maximum likelihood estimator (NPMLE) for the mixing distribution, introduced by Kiefer and Wolfowitz [KW56], plays a central role in this framework. Laird [Lai78] developed practical algorithms for computing the NPMLE, and Lindsay [Lin83] established that it is a discrete measure with at most n support points. More recently, Polyanskiy and Wu [PW20] discovered a remarkable self-regularization property of the NPMLE: even without explicit regularization, the NPMLE for subgaussian mixtures has $O(\log n)$ support points with high probability. The REBayes package [KM14, KG17] provides efficient implementations of NPMLE-based procedures.

A major breakthrough in the theoretical study of empirical Bayes was made by Jiang and Zhang [JZ09] in the Gaussian compound decision problem. This work inspires a flurry of subsequent studies in Gaussian [BG09, SG20, PW21, GIKL25] and Poisson models [BGR13, PW21, SW22, JPTW23, JPW25, HNWSW25]. We also refer to a recent book [Efr24] for an overview of empirical Bayes approaches.

Our approach differs from classical empirical Bayes in an important way: while standard empirical Bayes treats the heterogeneous parameters $(\sigma_1, \dots, \sigma_n)$ as the primary estimation target and applies Bayes rules, we treat them as nuisance parameters and focus on estimating the common mean μ through a profile likelihood [MVdV00, Sev00]. This perspective connects our work to the literature on profile likelihood inference with infinite-dimensional nuisance parameters [MVdV97, She97]. Nevertheless, our approach is still based on an important idea of empirical Bayes, where we replace heterogeneity by a mixing distribution and learn it from data.

Density estimation and sieve MLE theory. The analysis of our estimator relies on density estimation theory for the MLE in mixture models. The foundational work of Wong and Shen [WS95] established probability inequalities for likelihood ratios and convergence rates for sieve MLEs, showing that the rate is governed by the Hellinger metric entropy of the density class. Van de Geer [vdG00]

developed a comprehensive theory of M-estimation with empirical processes. This similar idea is used in our entropic upper bound in [Lemma 2](#).

Specializing to the MLE in mixture models, its density estimation performance under the Hellinger distance has been studied in both normal location mixture [[GvdV01](#), [GvdV07](#), [Zha09](#)], normal scale mixture [[GvdV01](#), [GvdV07](#), [IS25](#)], and Poisson mixture [[SW22](#), [JPW25](#)] models. Existing approaches are mainly based on the idea of (local) moment matching and polynomial approximations [[BM98](#), [GW00](#)], usually leading to $\text{polylog}(\frac{1}{\varepsilon})$ dependence on the radius ε in the metric entropy bound. However, the existing dependence on $\frac{\sigma_{\max}}{\sigma_{\min}}$ in the normal scale mixture is not tight, and our analysis improves the dependence to logarithmic using generalized method of moments and Chebyshev polynomial approximations, which better capture the analytic structure of the Gaussian kernel. This improvement is crucial for obtaining minimax-optimal rates that are robust to the ratio $\frac{\sigma_{\max}}{\sigma_{\min}}$.

Modulus of continuity and minimax lower bounds. Our upper bounds are expressed in terms of the Hellinger modulus of continuity of the location family, following the geometric approach to minimax theory developed by Donoho and Liu [[DL91](#)]. Le Cam’s two-point method [[LC73](#), [LC86](#)] provides matching lower bounds: if $H^2(f_{\mu_1, G}, f_{\mu_2, G}) \leq 1/n$, then one cannot reliably distinguish μ_1 from μ_2 , yielding a lower bound of $\omega_{H^2, G}(1/n)$ for estimating μ . Therefore, [Theorem 1.1](#) can be viewed as a duality result to Le Cam’s lower bound; such duality has appeared in the literature for linear functionals [[DL91](#), [JN09](#), [PW19](#)] and more recently for mean estimation in location families [[CV25](#)]. There the mixing distribution G_n is known, and the estimator is of a different Birgé–Le Cam type. In comparison, our empirical Bayes perspective shows that the profile MLE attains a comparable bound even when G_n is unknown.

Computation. The optimization problem in [\(3\)](#) is nonconvex due to the joint optimization over (μ, G) , but for fixed μ reduces to a convex problem in G . Algorithms for computing the NPMLE in mixture models include the EM algorithm [[DLR77](#), [Lai78](#)], interior point methods [[LK92](#)], and the Frank–Wolfe algorithm [[FW56](#), [Jag13](#)]. Koenker and Mizera [[KM14](#)] reformulated the NPMLE computation as a convex optimization problem, enabling efficient solutions for large-scale problems. Our implementation uses a successive maximization strategy that alternates between Frank–Wolfe updates for G and line search for μ ; similar block coordinate approaches are common in semiparametric estimation [[MVDVRS99](#), [VdV98](#)].

B Proof of [Theorem 1.1](#)

To complete the proof of [Theorem 1.1](#), we prove a symmetrization inequality which reduces possibly different scale mixtures to a single mixture.

Lemma 8. *Let $\mu_1, \mu_2 \in \mathbb{R}$, and G_1, G_2 be two prior distributions over $[0, \infty)$. Then*

$$H^2(f_{\mu_1, G_1}, f_{\mu_2, G_2}) \geq \frac{1}{4} H^2(f_{|\mu_1 - \mu_2|, G_1}, f_{-|\mu_1 - \mu_2|, G_1}).$$

The proof of [Theorem 1.1](#) follows by $(\mu_1, G_1) = (\mu, G_n)$ and $(\mu_2, G_2) = (\hat{\mu}, \hat{G})$ in [Lemma 8](#) and [Theorem 1.3](#). Crucially, thanks to [Lemma 8](#), the prior G_2 , which corresponds to the estimated prior \hat{G} , does not appear on the right-hand side and is therefore not subject to any structural requirements such as those imposed on G_1 . This asymmetry is the key technical reason why our estimator adapts to the signal size m in the subset-of-signals problem.

Proof of Lemma 8. By translation and reflection invariance, we may assume that $\mu_1 = \mu \geq 0$, and $\mu_2 = 0$. The proof will rely on an easy symmetric relationship:

$$f_{\mu,G}(x) = \mathbb{E}_{\sigma \sim G} \left[\frac{1}{\sigma} \varphi \left(\frac{x - \mu}{\sigma} \right) \right] = \mathbb{E}_{\sigma \sim G} \left[\frac{1}{\sigma} \varphi \left(\frac{-x + \mu}{\sigma} \right) \right] = f_{-\mu,G}(-x). \quad (15)$$

We repeatedly apply this symmetry to get

$$\begin{aligned} H^2(f_{\mu,G_1}, f_{0,G_2}) &= \int_{-\infty}^{\infty} \left(\sqrt{f_{\mu,G_1}(x)} - \sqrt{f_{0,G_2}(x)} \right)^2 dx \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \left[\left(\sqrt{f_{\mu,G_1}(x)} - \sqrt{f_{0,G_2}(x)} \right)^2 + \left(\sqrt{f_{\mu,G_1}(-x)} - \sqrt{f_{0,G_2}(-x)} \right)^2 \right] dx \\ &\stackrel{(15)}{=} \frac{1}{2} \int_{-\infty}^{\infty} \left[\left(\sqrt{f_{\mu,G_1}(x)} - \sqrt{f_{0,G_2}(x)} \right)^2 + \left(\sqrt{f_{\mu,G_1}(-x)} - \sqrt{f_{0,G_2}(x)} \right)^2 \right] dx \\ &\geq \frac{1}{4} \int_{-\infty}^{\infty} \left(\sqrt{f_{\mu,G_1}(x)} - \sqrt{f_{\mu,G_1}(-x)} \right)^2 dx \\ &\stackrel{(15)}{=} \frac{1}{4} \int_{-\infty}^{\infty} \left(\sqrt{f_{\mu,G_1}(x)} - \sqrt{f_{-\mu,G_1}(x)} \right)^2 dx \\ &= \frac{1}{4} H^2(f_{\mu,G_1}, f_{-\mu,G_1}), \end{aligned}$$

and the inequality step uses $a^2 + b^2 \geq \frac{(a-b)^2}{2}$. This is the desired result. \square

C Density Estimation: Proof of Theorem 1.3

In this section, we provide the proofs of the key lemmas used to establish Theorem 1.3.

C.1 Proof of Lemma 2

Our proof mostly follows the arguments in [vdG00]. We first mimic the arguments of [vdG00, Lemma 4.1]. Let P_n be the empirical distribution of X_1, \dots, X_n , then

$$\begin{aligned} -\frac{1}{n} \log \frac{1}{\beta} &\stackrel{(a)}{\leq} \int \log \frac{\hat{P}}{\bar{P}} dP_n \stackrel{(b)}{\leq} \frac{1}{2} \int \log \frac{\hat{P} + \bar{P}}{2\bar{P}} dP_n \\ &= \frac{1}{2} \int \log \frac{\hat{P} + \bar{P}}{2\bar{P}} d(P_n - \bar{P}) - \frac{1}{2} \text{KL} \left(\bar{P} \left\| \frac{\hat{P} + \bar{P}}{2} \right. \right) \\ &\stackrel{(c)}{\leq} \frac{1}{2} \int \log \frac{\hat{P} + \bar{P}}{2\bar{P}} d(P_n - \bar{P}) - \frac{1}{2} H^2 \left(\bar{P}, \frac{\hat{P} + \bar{P}}{2} \right), \end{aligned}$$

where (a) follows from the definition of \hat{P} and $\bar{P} \in \mathcal{P}$, (b) is due to the concavity of $x \mapsto \log x$, and (c) uses the inequality $\text{KL} \geq H^2$. Therefore,

$$H^2(\hat{P}, \bar{P}) \stackrel{(d)}{\leq} 16 H^2 \left(\bar{P}, \frac{\hat{P} + \bar{P}}{2} \right) \leq 16 \int \log \frac{\hat{P} + \bar{P}}{2\bar{P}} d(P_n - \bar{P}) + \frac{32}{n} \log \frac{1}{\beta}, \quad (16)$$

where (d) follows from [vdG00, Lemma 4.2]. This is called the “basic inequality” in [vdG00].

The next step is to prove a high-probability upper bound on the integral in (16), using empirical processes. For $P \in \mathcal{P}$, let $Z_i(P) = \frac{1}{2} \log \frac{P + \bar{P}}{2\bar{P}}(X_i)$. For the subexponential norm $\rho^2(g) := 2\mathbb{E}[e^{|g|} - 1 - |g|]$, it holds that

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \rho^2(Z_i(P)) &\stackrel{(e)}{\leq} \frac{1}{n} \sum_{i=1}^n 8\mathbb{E} \left[\left(\left(\frac{P + \bar{P}}{2\bar{P}}(X_i) \right)^{1/2} - 1 \right)^2 \right] \\ &= 8\mathbb{E}_{\bar{P}} \left[\left(\left(\frac{P + \bar{P}}{2\bar{P}}(X_i) \right)^{1/2} - 1 \right)^2 \right] = 8H^2 \left(\frac{P + \bar{P}}{2}, \bar{P} \right) \leq 4H^2(P, \bar{P}), \end{aligned}$$

where (e) uses [vdG00, Lemma 7.1] and that $Z_i(P) \geq -\frac{1}{2} \log 2$. In addition, for any bracket $[P^L, P^U]$, we have $Z_i(P^L) \leq Z_i(P^U)$, and the similar arguments to [vdG00, Lemma 7.3] yield

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \rho^2(Z_i(P^U) - Z_i(P^L)) &\leq \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[\left(\left(\frac{P^U + \bar{P}}{P^L + \bar{P}}(X_i) \right)^{1/2} - 1 \right)^2 \right] \\ &= \mathbb{E}_{\bar{P}} \left[\left(\left(\frac{P^U + \bar{P}}{P^L + \bar{P}}(X_i) \right)^{1/2} - 1 \right)^2 \right] \\ &\leq 2H^2 \left(\frac{P^U + \bar{P}}{2}, \frac{P^L + \bar{P}}{2} \right) \leq H^2(P^U, P^L). \end{aligned}$$

Therefore, compared with the Hellinger bracketing number $N_{[]}(\varepsilon, \mathcal{P}(\delta), H)$, the conditions of [vdG00, Definition 8.1] are fulfilled with $(\delta, R) = (\varepsilon, 2\delta)$ and $F^c = \emptyset$. Since

$$\frac{1}{n} \sum_{i=1}^n Z_i(P) - \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n Z_i(P) \right] = \frac{1}{2} \int \log \frac{P + \bar{P}}{2\bar{P}} d(P_n - \bar{P})$$

coincides with the integral in the basic inequality (16), now [vdG00, Theorem 8.13] (with $R = 2^{s+1}\delta$ and $a = c2^{2s}\delta^2\sqrt{n}$) yields

$$\mathbb{P} \left(\exists P \in \mathcal{P} : \left| \int \log \frac{P + \bar{P}}{2\bar{P}} d(P_n - \bar{P}) \right| \geq c2^{2s}\delta^2 \wedge H(P, \bar{P}) \leq 2^{s+1}\delta \right) \leq C \exp \left(- \frac{n(2^s\delta)^2}{C^2} \right)$$

for every $s \geq 0$ with $2^s\delta \leq 1$. Finally, by summing over $s = 0, 1, \dots$, a standard peeling argument similar to [vdG00, Theorem 7.4] completes the proof.

C.2 Proof of Theorem 2.1

In this section, we prove Theorem 2.1 based on Lemma 3. The proof decomposes into three steps.

C.2.1 L_∞ covering for scale mixtures

Our first step is to establish an upper bound on the covering number of *pure* scale mixtures

$$\mathcal{P}_0 = \{f_{0,G} : \text{supp}(G) \subseteq [\sigma_{\min}, \sigma_{\max}]\}$$

under the L_∞ norm.

Lemma 9. For $\varepsilon \in (0, 1/2)$ and $\sigma_{\min} \leq 1$,

$$\log N(\varepsilon, \mathcal{P}_0, L_\infty) \leq C \log^3 \left(\frac{1}{\varepsilon \sigma_{\min}} \right) \log^4 \log \left(\frac{1}{\varepsilon \sigma_{\min}} \right).$$

Proof. The proof relies on [Lemma 3](#), with a suitable reparametrization and truncation. Let $t = 1/\sigma$ and H be the pushforward measure of G under $\sigma \mapsto 1/\sigma$, then H is supported on $[0, \sigma_{\min}^{-1}]$, and

$$f_{0,G}(x) = \mathbb{E}_{\sigma \sim G} \left[\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) \right] = \mathbb{E}_{t \sim H} \left[\frac{t}{\sqrt{2\pi}} \exp\left(-\frac{t^2 x^2}{2}\right) \right] =: \tilde{f}_H(x).$$

To construct an ε -cover of $\{\tilde{f}_H : \text{supp}(H) \subseteq [0, \sigma_{\min}^{-1}]\}$, we invoke [Lemma 3](#) with parameters

$$t_{\min} = \frac{\varepsilon}{4}, \quad t_{\max} = \frac{1}{\sigma_{\min}}, \quad x_{\min} = c_0(\varepsilon\sigma_{\min}^3)^{1/2}, \quad x_{\max} = C_0\varepsilon^{-1}\sqrt{\log(1/\varepsilon)}, \quad (17)$$

where $c_0, C_0 > 0$ are appropriate universal constants. By [Lemma 3](#), there exists a cover \mathcal{H} with $|\mathcal{H}| = \exp(O(\log^3 \frac{1}{\varepsilon\sigma_{\min}} \log^4 \log \frac{1}{\varepsilon\sigma_{\min}}))$ such that for every H supported on $[t_{\min}, t_{\max}]$, there exists $H_0 \in \mathcal{H}$ such that

$$\sup_{x \in [x_{\min}, x_{\max}]} \left| \tilde{f}_H(x) - \tilde{f}_{H_0}(x) \right| \leq \frac{\varepsilon}{4}.$$

We show that a similar inequality still holds even if $x \notin [x_{\min}, x_{\max}]$:

1. If $x \geq x_{\max}$, the function $t \mapsto te^{-t^2 x^2/2}$ is decreasing for $x \geq 1/t_{\min}$, hence

$$0 \leq \tilde{f}_H(x) \leq \frac{t_{\min}}{\sqrt{2\pi}} \exp\left(-\frac{t_{\min}^2 x^2}{2}\right) \leq \frac{\varepsilon}{4}$$

for a large constant C_0 . The same bound also holds for \tilde{f}_{H_0} , so that $|\tilde{f}_H(x) - \tilde{f}_{H_0}(x)| \leq \frac{\varepsilon}{4}$.

2. If $0 \leq x \leq x_{\min}$, note that

$$|\tilde{f}'_H(x)| = x \mathbb{E}_{t \sim H} \left[\frac{t^3}{\sqrt{2\pi}} \exp\left(-\frac{t^2 x^2}{2}\right) \right] \leq \frac{x_{\min}}{\sqrt{2\pi}\sigma_{\min}^3}.$$

Therefore, for a small constant c_0 ,

$$\begin{aligned} & |\tilde{f}_H(x) - \tilde{f}_{H_0}(x)| \\ & \leq |\tilde{f}_H(x) - \tilde{f}_H(x_{\min})| + |\tilde{f}_H(x_{\min}) - \tilde{f}_{H_0}(x_{\min})| + |\tilde{f}_{H_0}(x_{\min}) - \tilde{f}_{H_0}(x)| \\ & \leq 2 \cdot \frac{x_{\min}^2}{\sqrt{2\pi}\sigma_{\min}^3} + \frac{\varepsilon}{4} \leq \frac{\varepsilon}{2}. \end{aligned}$$

3. If $x \leq 0$, since $\tilde{f}_H(x)$ is an even function, the statement follows from symmetry.

Therefore, we have established that if $\text{supp}(H) \subseteq [t_{\min}, t_{\max}]$, then $\|\tilde{f}_H - \tilde{f}_{H_0}\|_{\infty} \leq \frac{\varepsilon}{2}$.

Next we extend to the case where $\text{supp}(H)$ could be $[0, t_{\max}]$. Note that if $\text{supp}(H) \subseteq [0, t_{\min}]$, then $\tilde{f}_H(x) \leq t_{\min} = \frac{\varepsilon}{4}$. Therefore, we can modify the cover \mathcal{H} by a new cover

$$\mathcal{H}' = \left\{ (1-w)\delta_0 + wH_0 : H_0 \in \mathcal{H}, w \in \{0, \delta, 2\delta, \dots, 1\} \right\} \quad (18)$$

with $\delta = \frac{\varepsilon}{4t_{\max}}$, so that $|\mathcal{H}'| = O(|\mathcal{H}|/\delta)$. Now for any H supported on $[0, t_{\max}]$, we write $H = (1-w)H_1 + wH_2$ with $\text{supp}(H_1) \subseteq [0, t_{\min}]$, $\text{supp}(H_2) \subseteq [t_{\min}, t_{\max}]$, and $w \geq 0$. Then for $H' = (1-w')\delta_0 + w'H_0 \in \mathcal{H}'$ with $|w - w'| \leq \delta$ and H_2 approximated by $H_0 \in \mathcal{H}$, we get

$$\begin{aligned} \|\tilde{f}_H - \tilde{f}_{H'}\|_{\infty} &= \|(1-w)\tilde{f}_{H_1} + w\tilde{f}_{H_2} - w'\tilde{f}_{H_0}\|_{\infty} \\ &\leq \|\tilde{f}_{H_1}\|_{\infty} + \|\tilde{f}_{H_2} - \tilde{f}_{H_0}\|_{\infty} + |w - w'| \cdot \|\tilde{f}_{H_0}\|_{\infty} \\ &\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \delta \frac{t_{\max}}{\sqrt{2\pi}} \leq \varepsilon. \end{aligned}$$

This shows that $\{\tilde{f}_{H'} : H' \in \mathcal{H}'\}$ is an ε -cover of $\{\tilde{f}_H : \text{supp}(H) \subseteq [0, \sigma_{\min}^{-1}]\}$ under L_∞ , with

$$\log |\mathcal{H}'| = \log |\mathcal{H}| + O\left(\log \frac{1}{\delta}\right) = O\left(\log^3 \frac{1}{\varepsilon \sigma_{\min}} \log^4 \log \frac{1}{\varepsilon \sigma_{\min}}\right).$$

□

C.2.2 Hellinger bracketing for scale mixtures

For the class of scale mixtures \mathcal{P}_0 , by the standard approach in [GvdV01], we can transform an L_∞ -cover into a Hellinger bracket.

Lemma 10. For $\varepsilon \in (0, 1/2)$,

$$\log N_{[]}(\varepsilon, \mathcal{P}_0, H) \leq C \log^3 \left(\frac{\sigma_{\max}}{\varepsilon \sigma_{\min}}\right) \log^4 \log \left(\frac{\sigma_{\max}}{\varepsilon \sigma_{\min}}\right).$$

Proof. Since the bracketing number is invariant by scaling $(\sigma_{\min}, \sigma_{\max})$ simultaneously, WLOG we may assume that $\sigma_{\min} \leq 1$ so that Lemma 9 can be applied. First, note that for (not necessarily probability) measures μ and ν ,

$$H^2(\mu, \nu) = \int (\sqrt{d\mu} - \sqrt{d\nu})^2 \leq \int |d\mu - d\nu| = \|\mu - \nu\|_1.$$

Therefore, $\log N_{[]}(\varepsilon, \mathcal{P}_0, H) \leq \log N_{[]}(\varepsilon^2, \mathcal{P}_0, L_1)$. To construct a L_1 -bracket, let p_1, \dots, p_N be an η -cover of \mathcal{P}_0 in L_∞ , with $\eta > 0$ to be specified later. We then construct the brackets $\{[P_i^L, P_i^U] : i \in [N]\}$ by $P_i^L := \max\{p_i - \eta, 0\}$, $P_i^U := \min\{p_i + \eta, R\}$, with $R(x) = \frac{1}{\sigma_{\min}} \exp(-\frac{x^2}{2\sigma_{\max}^2})$. Since every $p \in \mathcal{P}_0$ satisfies

$$p(x) = \mathbb{E}_{\sigma \sim G} \left[\frac{1}{\sigma} \exp(-\frac{x^2}{2\sigma^2}) \right] \leq \frac{1}{\sigma_{\min}} \exp(-\frac{x^2}{2\sigma_{\max}^2}) = R(x),$$

the inequality $\|p - p_i\|_\infty \leq \eta$ indeed implies $P_i^L \leq p \leq P_i^U$. To upper bound the L_1 norm, for every $B \geq 0$ we have

$$\int_{\mathbb{R}} |P_i^L(x) - P_i^U(x)| dx \leq \int_{|x| \geq B} R(x) dx + 2\eta B \leq \frac{2\sigma_{\max}^2}{B\sigma_{\min}} \exp(-\frac{B^2}{2\sigma_{\max}^2}) + 2\eta B,$$

where the last inequality uses the Mills' ratio upper bound. Therefore, by choosing

$$B = C_0 \sigma_{\max} \sqrt{\log\left(\frac{\sigma_{\max}}{\sigma_{\min} \varepsilon^2}\right)}, \quad \eta = \frac{\varepsilon^2}{C_0 B}$$

with a large universal constant $C_0 > 0$, we ensure that $\|P_i^U - P_i^L\|_1 \leq \varepsilon^2$. Finally, by Lemma 9, we can take $\log N = O(\log^3(\frac{1}{\sigma_{\min} \eta}) \log^4 \log(\frac{1}{\sigma_{\min} \eta})) = O(\log^3(\frac{\sigma_{\max}}{\sigma_{\min} \varepsilon}) \log^4 \log(\frac{\sigma_{\max}}{\sigma_{\min} \varepsilon}))$. □

C.2.3 Local Hellinger bracketing for location-scale mixtures

Finally, we extend the Hellinger bracketing for the smaller family \mathcal{P}_0 in Lemma 10 to a *local* Hellinger bracketing for the entire family $\mathcal{P} = \{f_{\mu, G} : \mu \in \mathbb{R}, \text{supp}(G) \subseteq [\sigma_{\min}, \sigma_{\max}]\}$ involving the location parameter in Theorem 2.1.

Proof of Theorem 2.1. Let $\bar{P} = f_{\mu_0, G_0} \in \mathcal{P}$ be the center of the local ball; by translation invariance we may assume that $\mu_0 = 0$. We first establish an upper bound on $|\mu|$ for every $f_{\mu, G} \in \mathcal{P}(\bar{P}, \delta)$. In fact, by the symmetrization inequality in Lemma 8, we have

$$\delta \geq H(f_{0, G_0}, f_{\mu, G}) \geq \frac{1}{2} H(f_{\mu, G_0}, f_{-\mu, G_0}).$$

By the variational lower bound of the Hellinger distance in (26), we get

$$H^2(f_{\mu, G_0}, f_{-\mu, G_0}) \geq \frac{1}{4} (f_{\mu, G_0}([0, \infty)) - f_{-\mu, G_0}([0, \infty)))^2 \geq \frac{1}{4} \left(2\Phi\left(\frac{|\mu|}{\sigma_{\max}}\right) - 1 \right)^2,$$

where Φ is the CDF of $\mathcal{N}(0, 1)$. For $\delta \leq 1/8$, solving $|\mu|$ from the above two inequalities yields $|\mu| \leq C\sigma_{\max}$ for a universal constant $C > 0$, and therefore

$$\mathcal{P}(\bar{P}, \delta) \subseteq \{f_{\mu, G} : |\mu| \leq C\sigma_{\max}, \text{supp}(G) \subseteq [\sigma_{\min}, \sigma_{\max}]\} =: \mathcal{P}_{\text{loc}}.$$

To construct a Hellinger bracket for \mathcal{P}_{loc} , we start from the Hellinger bracket $\{[P_i^L, P_i^U] : i \in [N]\}$ for \mathcal{P}_0 constructed in Lemma 10, with $\|P_i^U - P_i^L\|_1 \leq \frac{\varepsilon^2}{4}$. Let $\eta > 0$ be a parameter to be specified later, and $\{\mu_1, \dots, \mu_M\}$ be a uniform discretization of $[-C\sigma_{\max}, C\sigma_{\max}]$ with spacing 2η . Here $M = O(\frac{\sigma_{\max}}{\eta})$, and for $i \in [N], j \in [M]$, define

$$P_{i,j}^L(x) = \inf_{|z| \leq \eta} P_i^L(x - \mu_j - z), \quad P_{i,j}^U(x) = \sup_{|z| \leq \eta} P_i^U(x - \mu_j - z).$$

We claim that $\{[P_{i,j}^L, P_{i,j}^U] : i \in [N], j \in [M]\}$ is a bracket for \mathcal{P}_{loc} . In fact, for any $f_{\mu, G} \in \mathcal{P}_{\text{loc}}$, assume $P_i^L \leq f_{0, G} \leq P_i^U$ and $|\mu - \mu_j| \leq \eta$. For this pair (i, j) , we have

$$f_{\mu, G}(x) = f_{0, G}(x - \mu) \geq \inf_{|z| \leq \eta} f_{0, G}(x - \mu_j - \eta) \geq \inf_{|z| \leq \eta} P_i^L(x - \mu_j - \eta) = P_{i,j}^L(x),$$

and similarly $f_{\mu, G} \leq P_{i,j}^U$. To upper bound the L_1 norm of this bracket, recall that $P_i^L = \max\{p_i - \eta', 0\}$ for some $p_i \in \mathcal{P}_0$ and $\eta' > 0$ in the proof of Lemma 10. Therefore,

$$\begin{aligned} \sup_{|z| \leq \eta} P_i^L(x - \mu_j - z) - \inf_{|z| \leq \eta} P_i^L(x - \mu_j - z) &\leq 2\eta \cdot \sup_{|z| \leq \eta} |p'(x - \mu_j - z)| \\ &\leq \frac{2\eta}{\sqrt{2\pi}\sigma_{\min}^2} \exp\left(-\frac{(|x - \mu_j| - \eta)_+^2}{2\sigma_{\max}^2}\right), \end{aligned}$$

where the last step follows from simple algebra, with $x_+ = \max\{x, 0\}$. Consequently,

$$\begin{aligned} \left\| \sup_{|z| \leq \eta} P_i^L(\cdot - \mu_j - z) - \inf_{|z| \leq \eta} P_i^L(\cdot - \mu_j - z) \right\|_1 &\leq \frac{2\eta}{\sqrt{2\pi}\sigma_{\min}^2} \int_{\mathbb{R}} \exp\left(-\frac{(|x - \mu_j| - \eta)_+^2}{2\sigma_{\max}^2}\right) dx \\ &\leq \frac{C\eta}{\sigma_{\min}^2} (\eta + \sigma_{\max}). \end{aligned} \tag{19}$$

On the other hand, since both functions P_i^U and P_i^L constructed in the proof of Lemma 10 are even and non-increasing on $[0, \infty)$, we have

$$\sup_{|z| \leq \eta} P_i^U(x - \mu_j - z) = P_i^U((|x - \mu_j| - \eta)_+),$$

and similarly for P_i^L . Therefore, integrating separately over two regimes $|x - \mu_j| \leq \eta$ and $|x - \mu_j| > \eta$ yields

$$\begin{aligned} & \left\| \sup_{|z| \leq \eta} P_i^U(\cdot - \mu_j - z) - \sup_{|z| \leq \eta} P_i^L(\cdot - \mu_j - z) \right\|_1 \\ &= 2\eta |P_i^U(0) - P_i^L(0)| + \|P_i^U - P_i^L\|_1 \leq 4\eta\eta' + \frac{\varepsilon^2}{4} \leq \frac{\varepsilon^2}{2}, \end{aligned} \quad (20)$$

where the first inequality follows from $\|P_i^U - P_i^L\|_\infty \leq 2\eta'$ and $\|P_i^U - P_i^L\|_1 \leq \frac{\varepsilon^2}{4}$, and the second inequality follows from the choice of $\eta' \leq \frac{\varepsilon^2}{4}$ in the proof of [Lemma 10](#) and $\eta \leq \frac{1}{4}$. Now by the triangle inequality and (19), (20), we obtain $\|P_{i,j}^U - P_{i,j}^L\|_1 \leq \varepsilon^2$ as long as

$$\eta = c_0 \left(\varepsilon \sigma_{\min} \wedge \frac{\varepsilon^2 \sigma_{\min}^2}{\sigma_{\max}} \right)$$

for a small universal constant $c_0 > 0$. Since $H^2(\mu, \nu) \leq \|\mu - \nu\|_1$, this implies that $\{[P_{i,j}^L, P_{i,j}^U] : i \in [N], j \in [M]\}$ is an ε -Hellinger bracket with size at most

$$\exp \left(O \left(\log^3 \left(\frac{\sigma_{\max}}{\varepsilon \sigma_{\min}} \right) \log^4 \log \left(\frac{\sigma_{\max}}{\varepsilon \sigma_{\min}} \right) + \log \frac{\sigma_{\max}}{\eta} \right) \right) = \exp \left(O \left(\log^3 \left(\frac{\sigma_{\max}}{\varepsilon \sigma_{\min}} \right) \log^4 \log \left(\frac{\sigma_{\max}}{\varepsilon \sigma_{\min}} \right) \right) \right).$$

This completes the proof. \square

C.3 Proof of [Lemma 3](#)

From [Lemma 4](#), it suffices to find functions $a_1(t), \dots, a_L(t)$ and $g_1(x), \dots, g_L(x)$ such that $|a_k(t)| + t|a'_k(t)| \leq A$ for all $t \in [t_{\min}, t_{\max}]$, $|g_k(x)| \leq G$ for all $x \in [x_{\min}, x_{\max}]$, and

$$\sup_{x \in [x_{\min}, x_{\max}], t \in [t_{\min}, t_{\max}]} \left| t e^{-\frac{t^2 x^2}{2}} - \sum_{k=1}^L a_k(t) g_k(x) \right| \leq \varepsilon. \quad (21)$$

Recall from [Lemma 5](#), for the function $h(v) = e^{-K e^{\lambda v}}$ on $v \in [-1, 1]$, we have the approximation

$$|h(v) - P_L(v)| \leq \varepsilon' \quad (22)$$

for all $v \in [-1, 1]$ where $L = O(\log(1/\varepsilon') + \lambda \log(\lambda/\varepsilon'))$ and P_L is a polynomial of degree L in v . Recall that the degree- L Chebyshev approximation of h corresponds to the polynomial $P_L(v) = \sum_{j=0}^L c_j T_j(v)$, where $T_j(x)$ is the degree- j Chebyshev polynomial with $T_j(\cos(\theta)) = \cos(j\theta)$, and c_j is the Chebyshev coefficient of h defined as

$$c_0 = \frac{1}{\pi} \int_0^\pi h(\cos(\theta)) d\theta, \quad c_j = \frac{2}{\pi} \int_0^\pi h(\cos(\theta)) \cos(j\theta) d\theta, \quad j \geq 1. \quad (23)$$

Since T_j are polynomials in v , each of degree at most L , we can write $P_L(v) = \sum_{j=0}^L p_j v^j$. Now set $u := \log t + \log x$, so that $u \in [u_{\min}, u_{\max}]$ where $u_{\min} = \log(t_{\min} x_{\min})$ and $u_{\max} = \log(t_{\max} x_{\max})$. Define the affine rescaling

$$v(u) := \frac{2u - (u_{\min} + u_{\max})}{u_{\max} - u_{\min}} \in [-1, 1].$$

Then

$$e^{-\frac{t^2 x^2}{2}} = \exp\left(-\frac{1}{2}e^{2u}\right) = \exp\left(-Ke^{\lambda v(u)}\right),$$

where $K = \frac{1}{2}e^{u_{\min}+u_{\max}} = \frac{1}{2}t_{\min}t_{\max}x_{\min}x_{\max}$ and $\lambda = u_{\max} - u_{\min} = \log \frac{t_{\max}x_{\max}}{t_{\min}x_{\min}}$. Therefore, applying the Chebyshev approximation of $h(v) = e^{-Ke^{\lambda v}}$ and substituting $v = v(u)$ yields

$$\left|e^{-\frac{t^2 x^2}{2}} - \sum_{i=0}^L p_i v(u)^i\right| \leq \varepsilon' \quad (24)$$

for all $t \in [t_{\min}, t_{\max}]$ and $x \in [x_{\min}, x_{\max}]$. Since $v(u)$ is affine in u , we can expand $\sum_{i=0}^L p_i v(u)^i$ into a degree- L polynomial in u , i.e.,

$$\sum_{i=0}^L p_i v(u)^i = \sum_{i=0}^L \tilde{p}_i u^i,$$

and then expanding the term $(\log t + \log x)^i$ using the binomial theorem, we have

$$\left|te^{-\frac{t^2 x^2}{2}} - \sum_{i=0}^L \sum_{j=0}^i \tilde{p}_i \binom{i}{j} (\log t)^j (\log x)^{i-j} t\right| \leq t_{\max} \varepsilon'. \quad (25)$$

We are left to bound the size of the coefficients $\tilde{p}_i \binom{i}{j}$. Since, $v(u)$ is an affine change of variables, with coefficients $2/\lambda$ and $\log(2K)/\lambda$, we have using the binomial theorem $|\tilde{p}_i| \leq \max_i |p_i| L \cdot 2^{2L} ((\log(2K)/\lambda)^L \wedge 1)$. To bound p_i , first note that since $h(u) = e^{-Ke^{\lambda u}}$ is a bounded function on $[-1, 1]$, by (23), all Chebyshev coefficients have magnitude at most 2. Further, we have that the coefficients of the Chebyshev polynomial $T_k(u)$ are bounded by 2^{k-1} in absolute value (see e.g., [Tre19]). We also trivially have $\binom{i}{j} \leq 2^i \leq 2^L$. Finally, for the functions in the approximation, we have $|t(\log t)^j| \leq \max\{\log^L(1/t_{\min}), t_{\max} \log^L(t_{\max})\}$ and $|\log x|^{i-j} \leq \max\{\log^L(x_{\max}), \log^L(1/x_{\min})\}$ for all $t \in [t_{\min}, t_{\max}]$ and $x \in [x_{\min}, x_{\max}]$. Combining these three bounds, we have that an approximation of the form in (21) holds with bound on the size of the terms given by $A \cdot G \leq L2^{4L} t_{\max} (\log(t_{\max}/t_{\min}))^{2L} (\log(x_{\max}/x_{\min}))^{2L}$. Further, there are at most L^2 such terms in the approximation. Setting $\varepsilon' = \varepsilon/t_{\max}$, and plugging into Lemma 4, we conclude that the L_{∞} covering number of \mathcal{F} is bounded by

$$\begin{aligned} & \log N(\varepsilon, \mathcal{F}, L_{\infty}([x_{\min}, x_{\max}])) \\ & \leq O\left(L^2 \log \frac{AGL^2}{\varepsilon}\right) \\ & = O\left(L^2 \log \left(\frac{2^{2L} t_{\max}^2 (\log(t_{\max}/t_{\min}))^{2L} (\log(x_{\max}/x_{\min}))^{2L} L^3}{\varepsilon}\right)\right) \\ & = O\left(\log^3 \left(\frac{x_{\max} t_{\max}}{x_{\min} t_{\min} \varepsilon}\right) \cdot \log^4 \log \left(\frac{x_{\max} t_{\max}}{x_{\min} t_{\min} \varepsilon}\right)\right) \end{aligned}$$

as required.

C.4 Proof of Lemma 4

By Carathéodory's theorem, for any $H \in \mathcal{P}([t_{\min}, t_{\max}])$, there exists a distribution $H' \in \mathcal{P}([t_{\min}, t_{\max}])$ with at most $O(L)$ atoms such that $\mathbb{E}_{t \sim H}[a_k(t)] = \mathbb{E}_{t \sim H'}[a_k(t)]$ for all $k \in [L]$. By discretizing the

support of H' into a geometric grid $\{t_{\min}, e^\delta t_{\min}, \dots, t_{\max}\}$, and the weights of H' into a δ -net on the simplex under $\|\cdot\|_1$, with $\delta = \frac{\varepsilon}{2AGL}$, this forms a finite set \mathcal{H} of size $\exp(O(L \log \frac{AGL}{\varepsilon}))$. In addition, if $H' = \sum_{i=1}^m w_i \delta_{t_i}$ and $H'' = \sum_{i=1}^m w'_i \delta_{t'_i}$ with $\max_{i \in [m]} |\log t_i - \log t'_i| \leq \delta$ and $\|w - w'\|_1 \leq \delta$, we have

$$\begin{aligned} |\mathbb{E}_{t \sim H'}[a_k(t)] - \mathbb{E}_{t \sim H''}[a_k(t)]| &\leq \left| \sum_{i=1}^m w_i (a_k(t_i) - a_k(t'_i)) \right| + \left| \sum_{i=1}^m (w_i - w'_i) a_k(t'_i) \right| \\ &\leq \sum_{i=1}^m w_i A |\log t_i - \log t'_i| + \|w - w'\|_1 \cdot A \leq 2\delta A. \end{aligned}$$

Since $\delta = \frac{\varepsilon}{2AGL}$ and $|g_k(x)| \leq G$, we conclude that any $H' \in \mathcal{P}([t_{\min}, t_{\max}])$ with at most $O(L)$ atoms can be approximated by some $H'' \in \mathcal{H}$ such that

$$\sup_{x \in [x_{\min}, x_{\max}], t \in [t_{\min}, t_{\max}]} \left| \sum_{k=1}^L \mathbb{E}_{t \sim H'}[a_k(t)] g_k(x) - \sum_{k=1}^L \mathbb{E}_{t \sim H''}[a_k(t)] g_k(x) \right| \leq \varepsilon.$$

By triangle inequality, this finite set \mathcal{H} induces a 2ε -covering of \mathcal{F} .

D Bounding Modulus of Continuity

In this section, we establish upper bounds of the Hellinger modulus of continuity in (4), with various choices of G_n . Specifically, we prove the upper bound in [Lemma 1](#) for the subset-of-signals problem, and upper bounds in [Corollary 1.1](#) for explicit examples of G_n .

D.1 Modulus of Continuity for SoS Priors: Proof of Lemma 1

The proof of [Lemma 1](#) relies on the following variational lower bound of squared Hellinger distance:

$$H^2(P, Q) = \int \frac{(P - Q)^2}{(\sqrt{P} + \sqrt{Q})^2} \geq \frac{1}{2} \int \frac{(P - Q)^2}{P + Q} = \frac{1}{2} \sup_T \frac{(\mathbb{E}_P[T] - \mathbb{E}_Q[T])^2}{\mathbb{E}_P[T^2] + \mathbb{E}_Q[T^2]}, \quad (26)$$

where the supremum is over all test functions $T : \mathcal{X} \rightarrow \mathbb{R}$, and the last step is Cauchy–Schwarz. In the definition of Hellinger modulus of continuity, suppose the choices $P = f_{\mu, G}$ and $Q = f_{0, G}$ satisfy $H^2(P, Q) \leq t$. Now choosing test function $T_\Delta(x) = 1_{[\mu, \mu + \Delta]}(x)$, the variational form gives

$$\frac{\left[\mathbb{E}_G \left(\Phi \left(\frac{\Delta}{\sigma} \right) - \Phi(0) - \Phi \left(\frac{\Delta + \mu}{\sigma} \right) + \Phi \left(\frac{\mu}{\sigma} \right) \right) \right]^2}{\mathbb{E}_G \left(\Phi \left(\frac{\Delta}{\sigma} \right) - \Phi(0) + \Phi \left(\frac{\Delta + \mu}{\sigma} \right) - \Phi \left(\frac{\mu}{\sigma} \right) \right)} \leq 2t. \quad (27)$$

Here $\Phi(x) = \int_{-\infty}^x \varphi(t) dt$ denote the standard Gaussian CDF. Taking $\Delta = \infty$, and noting that the denominator always lies in $[0, 1]$, we obtain

$$\sqrt{2t} \geq \mathbb{E}_G \left(\Phi \left(\frac{\mu}{\sigma} \right) - \Phi(0) \right) \geq \Phi'(1) \cdot \mathbb{E}_G \left[\frac{\mu \mathbf{1}_{\{\sigma \geq \mu\}}}{\sigma} \right]$$

due to $\Phi(x) - \Phi(0) \geq \Phi'(1)x$ for $x \in [0, 1]$. This shows that for a universal $c_1 > 0$,

$$\mathbb{E}_G \left[\frac{\mathbf{1}_{\{\sigma \geq \mu\}}}{\sigma} \right] \leq \frac{c_1 \sqrt{t}}{\mu}. \quad (28)$$

Next we take $\Delta = 1$ in (27). For the denominator, we have

$$\begin{aligned} \mathbb{E}_G\left(\Phi\left(\frac{1}{\sigma}\right) - \Phi(0) + \Phi\left(\frac{1+\mu}{\sigma}\right) - \Phi\left(\frac{\mu}{\sigma}\right)\right) &\leq 2\mathbb{E}_G\left(\Phi\left(\frac{1}{\sigma}\right) - \Phi(0)\right) \\ &\leq 2\left(\frac{1}{2}G([0, 1]) + \Phi'(0) \cdot \mathbb{E}_G\left[\frac{\mathbf{1}_{\{\sigma \geq 1\}}}{\sigma}\right]\right) \\ &\leq G([0, 1]) + \mathbb{E}_G\left[\frac{\mathbf{1}_{\{\sigma \geq 1\}}}{\sigma}\right]. \end{aligned} \quad (29)$$

For the numerator, we distinguish into two cases.

Case I: $\mu \geq 1$. Let

$$\Xi_{\mu, \sigma} := \Phi\left(\frac{1}{\sigma}\right) - \Phi(0) - \Phi\left(\frac{1+\mu}{\sigma}\right) + \Phi\left(\frac{\mu}{\sigma}\right).$$

When $\mu \geq 1$, simple algebra yields that for a universal constant $c_2 > 0$,

$$\Xi_{\mu, \sigma} \geq c_2 \begin{cases} 1 & \text{if } \sigma < 1, \\ \frac{1}{\sigma} & \text{if } 1 \leq \sigma < \mu, \\ 0 & \text{if } \sigma \geq \mu. \end{cases} \quad (30)$$

Therefore,

$$\mathbb{E}_G[\Xi_{\mu, \sigma}] \geq c_2 \left(G([0, 1]) + \mathbb{E}_G\left[\frac{\mathbf{1}_{\{1 \leq \sigma \leq \mu\}}}{\sigma}\right] \right), \quad (31)$$

and

$$\begin{aligned} \left(G([0, 1]) + \mathbb{E}_G\left[\frac{\mathbf{1}_{\{1 \leq \sigma \leq \mu\}}}{\sigma}\right] \right)^2 &\stackrel{(a)}{\leq} c_3 t \left(G([0, 1]) + \mathbb{E}_G\left[\frac{\mathbf{1}_{\{\sigma \geq 1\}}}{\sigma}\right] \right) \\ &\stackrel{(b)}{\leq} c_3 t \left(G([0, 1]) + \mathbb{E}_G\left[\frac{\mathbf{1}_{\{1 \leq \sigma \leq \mu\}}}{\sigma}\right] + \frac{c_1 \sqrt{t}}{\mu} \right). \end{aligned}$$

Here (a) follows from (27) and (29), and (b) uses (28). Solving this quadratic inequality gives

$$G([0, 1]) + \mathbb{E}_G\left[\frac{\mathbf{1}_{\{1 \leq \sigma \leq \mu\}}}{\sigma}\right] \leq c_4 \left(t + \frac{t^{3/4}}{\mu^{1/2}} \right).$$

As $G([0, 1]) \geq p \geq Ct$ for $C \geq 2c_4$, this inequality gives that $\mu = O\left(\frac{t^{3/2}}{p^2}\right)$.

Case II: $0 \leq \mu < 1$. In this case, one can verify the following lower bounds for $\Xi_{\mu, \sigma}$:

$$\Xi_{\mu, \sigma} \geq c_2 \begin{cases} 1 & \text{if } \sigma < \mu, \\ \frac{\mu}{\sigma} & \text{if } \mu \leq \sigma < 1, \\ 0 & \text{if } \sigma \geq 1. \end{cases} \quad (32)$$

Therefore, the numerator is lower bounded as

$$\mathbb{E}_G[\Xi_{\mu, \sigma}] \geq c_2 \left(G([0, \mu]) + \mathbb{E}_G\left[\frac{\mu \mathbf{1}_{\{\mu \leq \sigma \leq 1\}}}{\sigma}\right] \right) \geq c_2 \mu \cdot G([0, 1]). \quad (33)$$

Consequently, by (27), (28), and (29), we get

$$\mu^2 G([0, 1])^2 \leq c_3 t \left(G([0, 1]) + \mathbb{E}_G \left[\frac{\mathbf{1}_{\{\sigma \geq 1\}}}{\sigma} \right] \right) \leq c_3 t \left(G([0, 1]) + \frac{c_1 \sqrt{t}}{\mu} \right),$$

and further solve this quadratic inequality to find

$$G([0, 1]) \leq c_4 \left(\frac{t}{\mu^2} + \frac{t^{3/4}}{\mu^{3/2}} \right).$$

Finally, as $G([0, 1]) \geq p$, we obtain $\mu = O\left(\frac{t^{1/2}}{p^{1/2}} + \frac{t^{1/2}}{p^{2/3}}\right) = O\left(\frac{t^{1/2}}{p^{2/3}}\right)$ as $p \leq 1$.

To reach the desired conclusion, note that when $p^{4/3} < t \leq \frac{p}{C}$, we have either $\mu = O\left(\frac{t^{3/2}}{p^2}\right)$ (Case I), or $\mu \leq 1$ (assumption of Case II). Since the upper bound in Case I is larger, we conclude that $\mu = O\left(\frac{t^{3/2}}{p^2}\right)$. Similarly, when $t \leq p^{4/3}$, we have either $\mu = O\left(\frac{t^{3/2}}{p^2}\right)$ (Case I), or $\mu = O\left(\frac{t^{1/2}}{p^{2/3}}\right)$ (Case II). Here the upper bound in Case II dominates, so that $\mu = O\left(\frac{t^{1/2}}{p^{2/3}}\right)$.

D.2 Modulus of Continuity for Selected Priors: Proof of Corollary 1.1

To prove Corollary 1.1, we invoke Theorem 1.1 and establish upper bounds of the Hellinger modulus of continuity for various choices of G_n .

Equal variance. When $\sigma_i \equiv 1$, we have $G_n = \delta_1$. In this case,

$$H^2(f_{\mu_1, G_n}, f_{\mu_2, G_n}) = 2 - 2 \exp\left(-\frac{(\mu_1 - \mu_2)^2}{8}\right),$$

so that $\omega_{H^2, G_n}(t) = O(\sqrt{t})$. Now plugging in $t = \tilde{O}\left(\frac{1}{n}\right)$ gives the result.

Quadratic variance. Consider any $\mu > 0$ such that (27) holds for all $\Delta \geq 0$, with $G = G_n$. Next we choose $\Delta = \mu$ in (27) and proceed in the same way as (29) to obtain

$$G_n([0, \mu])^2 \leq c_1 t \left(G_n([0, \mu]) + \mu \mathbb{E}_{G_n} \left[\frac{\mathbf{1}_{\{\sigma \geq \mu\}}}{\sigma} \right] \right).$$

For $t \leq c_2$ with a small enough constant $c_2 > 0$, the above inequality shows that $\mu > n$ is impossible. For $1 \leq \mu \leq n$, since $\mathbb{E}_{G_n} \left[\frac{\mathbf{1}_{\{\sigma \geq \mu\}}}{\sigma} \right] \leq \mathbb{E}_{G_n} \left[\frac{1}{\sigma} \right] = \frac{1}{n} \sum_{i=1}^n \frac{1}{i} = O\left(\frac{\log n}{n}\right)$ and $G_n([0, \mu]) \asymp \frac{\mu}{n}$, we can solve that $\mu = O(nt \log n)$. Combining with the remaining case $\mu \leq 1$ leads to the final upper bound

$$\omega_{H^2, G_n}(t) = O(nt \log n \vee 1), \quad \text{if } t \leq c_2.$$

Plugging in $t = \tilde{O}\left(\frac{1}{n}\right)$ proves the target result.

Two variances. Again, consider some $\mu > 0$ such that (27) holds for every $\Delta \geq 0$, with $G = G_n = \frac{m}{n} \delta_1 + (1 - \frac{m}{n}) \delta_\sigma$. We consider two choices of Δ . First, choosing $\Delta = \sigma$ and upper bounding the denominator by 2 in (27), we get

$$\frac{n-m}{n} \left(\Phi(1) - \Phi(0) - \Phi\left(1 + \frac{\mu}{\sigma}\right) + \Phi\left(\frac{\mu}{\sigma}\right) \right) \leq c_1 \sqrt{t}.$$

Clearly, for $t \leq c_2$ with a small enough constant $c_2 > 0$, this inequality implies that $\mu \leq \sigma$. In this case, the left-hand side further scales as $\Omega(\frac{\mu}{\sigma})$, so that we get

$$\omega_{H^2, G_n}(t) = O(\sigma\sqrt{t}), \quad \text{if } t \leq c_2.$$

For $t = \tilde{O}(\frac{1}{n})$, this gives the general $O(\frac{\sigma}{\sqrt{n}})$ upper bound.

The second choice of Δ is $\Delta = 1$. If we upper bound the denominator of (27) by 2, then

$$\frac{m}{n} (\Phi(1) - \Phi(0) - \Phi(1 + \mu) + \Phi(\mu)) \leq c_1\sqrt{t}.$$

Therefore, if $\frac{n\sqrt{t}}{m} \leq c_2$ for a small enough constant $c_2 > 0$, we solve that $\mu = O(\frac{n\sqrt{t}}{m})$. For $t = \tilde{O}(\frac{1}{n})$, this proves the $\tilde{O}(\frac{\sqrt{n}}{m})$ upper bound for $m \geq \sqrt{n \log n}$. Instead, if we upper bound the denominator of (27) via (29), we obtain

$$\frac{m}{n} (\Phi(1) - \Phi(0) - \Phi(1 + \mu) + \Phi(\mu)) \leq c_1 \sqrt{t \left(\frac{m}{n} + \frac{1}{\sigma} \right)}.$$

For $t = \tilde{O}(\frac{1}{n})$ and $m \geq C(\sqrt{\frac{n \log n}{\sigma}} + \log n)$, it holds that

$$\frac{m}{n} \geq 10c_1 \sqrt{t \left(\frac{m}{n} + \frac{1}{\sigma} \right)}.$$

Solving the inequality gives the target upper bound $\mu = O(1)$.

α -mixture distributions. We use the upper bound in the two variances example. For $\alpha \in (0, 1)$, we use the upper bound $\tilde{O}(\frac{\sigma}{\sqrt{n}}) = \tilde{O}(n^{\alpha-1/2})$. For $\alpha \geq 1$, we note that for $m = c \log n$ with a large enough constant $c > 0$, it holds that

$$C \left(\sqrt{\frac{n \log n}{n^\alpha}} + \log n \right) \leq m < \sqrt{n \log n}.$$

Therefore, we can apply the $O(1)$ upper bound in this case.

E Additional Details of Section 4

E.1 Proof of Lemma 7

Since $X_i \neq 0$ for all $i \in [n]$, the function $G \mapsto f_{0,G}(X_i)$ has a finite upper bound depending only on X_i , so the log-likelihood $\sum_{i=1}^n \log f_{0,G}(X_i)$ cannot reach $+\infty$. Then standard compactness argument shows the existence of the NPMLE \hat{G} , and by the strict concavity of $x \mapsto \log x$, the vector of densities $(f_{0,\hat{G}}(X_1), \dots, f_{0,\hat{G}}(X_n))$ is unique. Denote it by (f_1, \dots, f_n) .

By the KKT condition (14), $\text{supp}(\hat{G})$ is a subset of the set of global maximizers of $\sigma \mapsto D_{\hat{G}}(\sigma)$. By differentiation, the set of critical points of $\sigma \mapsto D_{\hat{G}}(\sigma)$ is

$$C := \left\{ \sigma > 0 : \frac{1}{n} \sum_{i=1}^n \frac{f_i}{\sqrt{2\pi}} e^{-X_i^2/2\sigma^2} \left(\frac{X_i^2}{\sigma^2} - 1 \right) = 0 \right\}. \quad (34)$$

Clearly, $C \subseteq [\min_i |X_i|, \max_i |X_i|]$, so the same holds for $\text{supp}(\widehat{G})$. By combining repeated appearances of $|X_i|$, WLOG we assume that $|X_1|, \dots, |X_n|$ are distinct. Next we show that C is a finite set, with $|C| \leq 2n - 1$. Therefore, for some distinct constants c_1, c_2, \dots, c_n and a non-zero vector (d_1, \dots, d_{2n}) ,

$$\left\{ \frac{1}{\sigma^2} : \sigma \in C \right\} \subseteq \left\{ x > 0 : \sum_{i=1}^n (d_i + d_{n+i}x)e^{c_i x} = 0 \right\}.$$

To proceed, we recall the following result taken from [PS25, Page 48].

Lemma 11. *Let $c_1, \dots, c_n \in \mathbb{R}$ be distinct, and $P_i(x)$ be a polynomial in x with degree $m_i - 1$. Then the equation $\sum_{i=1}^n P_i(x)e^{c_i x} = 0$ has at most $(\sum_{i=1}^n m_i) - 1$ real solutions.*

By Lemma 11, the above equation has at most $2n - 1$ solutions. This shows that $|C| \leq 2n - 1$. Finally, since the continuous map $\sigma \mapsto D_{\widehat{G}}(\sigma)$ must have at least one critical point between two global maximizers, we conclude from the upper bound on $|C|$ that $|\text{supp}(\widehat{G})| \leq n$.

Finally we show the uniqueness of \widehat{G} . Since the map $\sigma \mapsto D_{\widehat{G}}(\sigma)$ only depends on \widehat{G} through the likelihood vector (f_1, \dots, f_n) , the set D of its global maximizers is a fixed set for all versions of \widehat{G} . In addition, $|D| \leq n$, so we can write $D = \{\sigma_1, \dots, \sigma_m\}$ with $m \leq n$, and any NPMLE \widehat{G} takes the form $\sum_{j=1}^m w_j \delta_{\sigma_j}$. It remains to show the uniqueness of (w_1, \dots, w_m) . Now by the uniqueness of (f_1, \dots, f_n) , (w_1, \dots, w_m) is a solution to the linear system

$$\sum_{j=1}^m w_j \frac{1}{\sigma_j} \varphi\left(\frac{X_i}{\sigma_j}\right) = f_{0, \widehat{G}}(X_i) = f_i, \quad \forall i \in [n].$$

To prove uniqueness, it suffices to show that the matrix $A = (A_{ij})_{i \in [n], j \in [m]}$ with $A_{ij} = \frac{1}{\sigma_j} \varphi\left(\frac{X_i}{\sigma_j}\right)$ has full column rank. By scaling the columns and taking $m \times m$ submatrices, it further suffices to proving that $B = (B_{ij}) \in \mathbb{R}^{m \times m}$ with $B_{ij} = \varphi\left(\frac{X_i}{\sigma_j}\right)$ has a full rank. Assuming the contrary, then $Bz = 0$ would have a non-zero solution $z \in \mathbb{R}^m$, and the map

$$t \mapsto \sum_{j=1}^m \exp\left(-\frac{t}{\sigma_j^2}\right) z_j$$

would have at least m distinct zeros. Since $\sigma_1, \dots, \sigma_m > 0$ are distinct, this is a contradiction to the statement of Lemma 11. This concludes the uniqueness of \widehat{G} .